

Algebraic Principles of Quantum Field Theory II

Quantum Coordinates and WDVV Equation

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Abstract. This paper is about algebro-geometrical structures on a moduli space \mathcal{M} of anomaly-free BV QFTs with finite number of inequivalent observables or in a finite superselection sector. We show that \mathcal{M} has the structure of F-manifold – a linear pencil of torsion-free flat connection with unity on the tangent space, in quantum coordinates. We study the notion of quantum coordinates for the family of QFTs, which determines the connection 1-form as well as every quantum correlation function of the family in terms of the 1-point functions of the initial theory. We then define free energy for an unital BV QFT and show that it is another avatar of morphism of QFT algebra. These results are consequences of the solvability of refined quantum master equation of the theory. We also introduce the notion of a QFT integral and study some properties of BV QFT equipped with a QFT integral. We show that BV QFT with a non-degenerate QFT integral leads to the WDVV equation—the formal Frobenius manifold structure on \mathcal{M} —if it admits a semi-classical solution of quantum master equation.

1. Introduction

This is the 2nd installment of the series of papers in a quest to find algebraic principles of general quantum field theory. A large part of this paper is a natural continuation of

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the last chapter of the previous one [1], dealing with the same class of quantum field theory called anomaly-free BV QFT with a finite number of equivalence classes of observables or in a finite superselection sector. We have shown that such a theory comes with its family parametrized by a formal smooth super (moduli) space \mathcal{M} with a base point, which corresponds to the initial QFT, in quantum coordinates. It was a natural consequence of the existence of solution to quantum master equation, which automatically gives a distinguished solution to the Batalin-Vilkovisky (BV) quantum master equation. Our quantum master equation was shown to govern the quantization of classical correlators such that its solution can be used to determine every quantum correlation functions from a generating set of 1-point functions.

In this paper we shall refine the previous notion of quantum master equation in its “up to homotopy” part and find and its solution. We shall also introduce a QFT integral, which is another piece of data possibly carried by a BV QFT. An immediate consequence shall be that the tangent space $T\mathcal{M}$ of the moduli space \mathcal{M} of a BV QFT with an QFT integral has certain algebraic structure, which, in special non-degenerate, semi-classical cases, reduces to that of K. Saito [2] and, independently, Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) [3,4], a.k.a. the Frobenius manifold structure formalized and studied in details by Dubrovin [5].

For us, the refined quantum master equation shall pave a way for understanding both morphisms of QFT algebras up to homotopy and homotopy path integrals, which are the subjects of the 3rd paper in this series [6]. We also isolate the notion of QFT complex from the definition of BV QFT algebra in [1], which is the core structure of general QFT algebras to appear in sequels.

The solvability of the refined quantum master equation implies that there is 3-tensor $A_{\alpha\beta}{}^\gamma$, which is a formal power series in an affine coordinates $t_H = \{t^\alpha\}$ of the graded space vector space H of equivalence class of observables, on \mathcal{M} satisfying

1. symmetry (graded commutativity and potentiality):

$$A_{\alpha\beta}{}^\gamma = (-1)^{|t^\alpha||t^\beta|} A_{\beta\alpha}{}^\gamma, \quad \frac{\partial A_{\beta\gamma}{}^\sigma}{\partial t^\alpha} - (-1)^{|t^\alpha||t^\beta|} \frac{\partial A_{\beta\gamma}{}^\sigma}{\partial t^\alpha} = 0,$$

2. relation (associativity):

$$\sum_{\rho} A_{\alpha\beta}{}^{\rho} A_{\rho\gamma}{}^{\sigma} = \sum_{\rho} A_{\beta\gamma}{}^{\rho} A_{\alpha\rho}{}^{\sigma},$$

3. unity: there is distinguished distinguish element t^0 such that $A_{0\beta}{}^\gamma = \delta_{\beta}{}^\gamma$,

4. homogeneity (Euler vector):

$$\sum_{\rho} |t^{\rho}| t^{\rho} \frac{\partial}{\partial t^{\rho}} A_{\alpha\beta}{}^{\gamma} = (|t^{\gamma}| - |t^{\beta}| - |t^{\alpha}|) A_{\alpha\beta}{}^{\gamma}.$$

We remark that a formal super-manifold with such a 3-tensor was called an F-manifold by Herling-Manin and studied in [7, 8]. The symmetry of the 3-tensor $A_{\alpha\beta}{}^{\gamma}$ implies that $A_{\alpha\beta}{}^{\gamma} = \frac{\partial^2 \Phi^{\gamma}}{\partial t^{\alpha} \partial t^{\beta}}$, for some $\{\Phi^{\gamma}\}$, so that the associativity relation and unity imply that

$$\begin{aligned} \sum_{\rho} \left(\frac{\partial^2 \Phi^{\rho}}{\partial t^{\alpha} \partial t^{\beta}} \right) \left(\frac{\partial^2 \Phi^{\sigma}}{\partial t^{\rho} \partial t^{\gamma}} \right) &= \sum_{\rho} \left(\frac{\partial^2 \Phi^{\rho}}{\partial t^{\beta} \partial t^{\gamma}} \right) \left(\frac{\partial^2 \Phi^{\sigma}}{\partial t^{\alpha} \partial t^{\rho}} \right), \\ \frac{\partial^2 \Phi^{\gamma}}{\partial t^0 \partial t^{\beta}} &= \delta_{\beta}{}^{\gamma}. \end{aligned} \quad (1.1)$$

The above equation may be viewed as a weaker version of WDDV equation, though its solution $\{\Phi^{\gamma}\}$ is not particularly relevant for us. Rather, the role of $A_{\alpha\beta}{}^{\gamma}$ as quantum correlation functions for the full family of BV QFT in quantum coordinates is our greater interest.

The coefficients of an expansion of $A_{\alpha\beta}{}^{\gamma}$ at $t_H = 0$:

$$A_{\alpha\beta}{}^{\gamma} = m_{\alpha\beta}{}^{\sigma} + \sum_{\rho} t^{\rho} m_{\rho\alpha\beta}{}^{\sigma} + \frac{1}{2!} \sum_{\rho_1, \rho_2} t^{\rho_2} t^{\rho_1} m_{\rho_1 \rho_2 \alpha\beta}{}^{\gamma} + \dots$$

correspond to the structure constants of the sequence m_2, m_3, m_4, \dots of graded symmetric products of ghost number zero on H , $m_n : S^n H \rightarrow H$, which were discussed in the previous paper and, together with the 1-point functions $\{\langle \mathbf{O}_{\alpha} \rangle\}$ and the QFT cycle, were used to determine all of the n -point quantum correlation function of the initial BV QFT. In this paper we shall show that, in fact, $A_{\alpha\beta}{}^{\gamma}$ can be used to determine all n -point correlation functions for every BV QFT in the family parametrized by \mathcal{M} . The key concept is that of quantum coordinates $\{T^{\gamma}\}$ for the family, which are formal power series in t_H and \hbar^{-1} involving the coefficients of $A_{\alpha\beta}{}^{\gamma}$ in the following form

$$\begin{aligned} T^{\gamma} &= t^{\gamma} - \frac{1}{2\hbar} \sum_{\alpha_1, \alpha_2} t^{\alpha_2} t^{\alpha_1} m_{\alpha_1 \alpha_2}{}^{\gamma} \\ &+ \frac{1}{6\hbar^2} \sum_{\alpha_1, \alpha_2, \alpha_3} t^{\alpha_3} t^{\alpha_2} t^{\alpha_1} \left(-\hbar m_{\alpha_1 \alpha_2 \alpha_3}{}^{\gamma} + \sum_{\rho} m_{\alpha_1 \alpha_2}{}^{\rho} m_{\rho \alpha_3}{}^{\gamma} \right) + \dots, \end{aligned}$$

such that

$$\sum_{\gamma} (-\hbar)^n \frac{\partial^n}{\partial t^{\alpha_1} \dots \partial t^{\alpha_n}} T^{\gamma} \langle \mathbf{O}_{\gamma} \rangle$$

are quantum correlation functions for the family. Conversely T^γ determine $A_{\alpha\beta}^\gamma$ as follows

$$A_{\alpha\beta}^\gamma = -\hbar \sum_{\rho} \frac{\partial \mathcal{G}_{\beta}^{\rho}}{\partial t^{\alpha}} \mathcal{G}_{\rho}^{-1\gamma},$$

where \mathcal{G} denotes the matrix with $\beta\gamma$ entry $\mathcal{G}_{\beta}^{\gamma} := \frac{\partial}{\partial t^{\beta}} T^{\gamma} = \delta_{\beta}^{\gamma} + \dots \in \mathbb{k}[[t_H, \hbar^{-1}]]$, which is invertible. Then (i) the graded commutativity of $A_{\alpha\beta}^\gamma$ is obvious (ii) both the potentiality and associativity reduces to $d\mathcal{G}^{-1} \wedge d\mathcal{G} = 0$, where $d = \sum_{\alpha} dt^{\alpha} \frac{\partial}{\partial t^{\alpha}}$ is the formal exterior derivative (iii) the unity becomes $\frac{\partial}{\partial t^0} \mathcal{G} = -\frac{1}{\hbar} \mathcal{G}$.

For an unital BV QFT, where the partition function $\langle 1 \rangle$ is normalizable to 1, free energy F can be defined as follows

$$e^{-F/\hbar} = 1 - \frac{1}{\hbar} \sum_{\gamma} T^{\gamma} \widehat{\langle \mathbf{O}_{\gamma} \rangle},$$

where $\widehat{\langle \mathbf{O}_{\alpha} \rangle}$ is normalized expectation value. Then, the free energy is a formal power series in t_H and \hbar and satisfies the following system of differential equations:

$$\hbar \frac{\partial^2 F}{\partial t^{\alpha} \partial t^{\beta}} = \left(\frac{\partial F}{\partial t^{\alpha}} \right) \left(\frac{\partial F}{\partial t^{\beta}} \right) - \sum_{\gamma} A_{\alpha\beta}^{\gamma} \left(\frac{\partial F}{\partial t^{\gamma}} \right),$$

where $-\hbar \frac{\partial F}{\partial t^{\alpha}} \big|_{t_H=0} = \widehat{\langle \mathbf{O}_{\alpha} \rangle}$. We shall see that the free energy is another avatar of a morphism of QFT algebra. There is also an exciting possibility to study phases of the moduli space \mathcal{M} via thermodynamical interpretation of the free energy. For this, however, we should be able to work with convergent power series instead of formal one.

In Sect. 4.4 of the previous paper [1] we have argued that the notion of quantum coordinates is a natural generalization of that of flat or special coordinates on moduli spaces of topological strings or conformal field theory in 2-dimensions [3, 4, 9, 10]. For the mathematics side, it was demonstrated that both the flat structure on the moduli space of universal unfolding of simple singularities due to K. Saito [2] and the flat coordinates on moduli space associated with differential BV algebra with $\partial\bar{\partial}$ -lemma due to Barannikov-Kontsevich [11] are also examples of quantum coordinates. We should, however, keep in mind that the associated moduli spaces of the above mentioned examples also come with a compatible flat metric $g_{\alpha\beta}$ on its tangent space:

$$g_{\alpha\beta} = (-1)^{|t^{\alpha}||t^{\beta}|} g_{\beta\alpha}, \quad \sum_{\rho} A_{\alpha\beta}^{\rho} g_{\rho\gamma} = \sum_{\rho} A_{\beta\gamma}^{\rho} g_{\alpha\rho}, \quad \frac{\partial g_{\beta\gamma}}{\partial t^{\alpha}} = 0,$$

which is invertible. The additional data $\{g_{\alpha\beta}\}$, then, implies that $A_{\alpha\beta\gamma} := \sum_{\rho} A_{\alpha\beta}^{\rho} g_{\rho\gamma}$ is totally graded symmetric for all 3-indices and there is some potential Φ such that

$A_{\alpha\beta\gamma} = \frac{\partial^3 \Phi}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$. Then the associativity and unity conditions become the WDVV equation:

$$\sum_{\mu,\nu} \left(\frac{\partial^3 \Phi}{\partial t^\alpha \partial t^\beta \partial t^\mu} \right) g^{\mu\nu} \left(\frac{\partial^3 \Phi}{\partial t^\nu \partial t^\gamma \partial t^\rho} \right) = \sum_{\mu,\nu} \left(\frac{\partial^3 \Phi}{\partial t^\beta \partial t^\gamma \partial t^\mu} \right) g^{\mu\nu} \left(\frac{\partial^3 \Phi}{\partial t^\alpha \partial t^\nu \partial t^\rho} \right),$$

$$g_{\alpha\beta} = \frac{\partial^2}{\partial t^\alpha \partial t^\beta} \left(\frac{\partial \Phi}{\partial t^0} \right),$$

where $g^{\alpha\beta}$ denote the inverse metric.

Our BV QFT package (BV QFT algebra plus QFT cycle) does not automatically lead to a compatible flat metric, while the above historical examples come with additional data such as suitable version of the Poincaré metric or the Barannikov-Kontsevich (BK) integral[11]. In this paper we introduce the notion of QFT integral, which supplies new quantum homotopy invariants in addition to those coming from a QFT cycle. A QFT integral shall be a QFT cycle with special property which can also be viewed as a generalization of BK integral. We, then, show that BV QFT with a QFT integral induces certain rich algebraic structure on $T\mathcal{M}$, which, in the semi-classical case, is exactly Frobenius manifold.

These results are further evidence of our assertion that the notion of quantum coordinates is a natural generalization of flat, or special, coordinates. The term quantum coordinates is motivated by the following phenomena: A solution to the quantum master equation automatically gives a solution to quantum descendant equation (BV quantum master equation) whose classical limit corresponds to a specific choice of a universal solution to the Maurer-Cartan equation (the classical BV master equation) governing the moduli space \mathcal{M} . Via such the solution the affine coordinates $t_H = \{t^\alpha\}$ on H give a distinguished coordinates called quantum coordinates on \mathcal{M} . In the semi-classical case such quantum coordinates has been called flat since $g_{\alpha\beta}$ happens to be independent of t_H .

We also note that the essential information of quantum correlations is summarized by the quantum coordinates $\{T(t_H)^\gamma\}$ for the family, which can be inverted after being regarded as a sequence of maps, parametrized by \hbar^{-1} , on H to a formal neighborhood around the base point in \mathcal{M} . Such inversion is reminiscent of the Abel-Jacobi inversion of period map on elliptic curve¹ as well as the mirror map and could be a universal problem associated with quantum field theory without anomaly. For us it

¹ It is interesting to note that an attempt to generalize this picture has led K. Saito to his original discovery of Frobenius manifold structure on a universal unfolding of singularity (see his own account in[12]).

adds strength to the mantra of our series that quantum field theory is the study of morphisms of QFT algebras, with quasi-isomorphisms being physical equivalences.

This paper is organized as follows. In Section 2 we setup notations and conventions as well as a streamlined summary of the the basic mathematical setting of the previous paper [1]. The notable additions are the definition of QFT complex and two propositions about it. The QFT complex is a core structure of general QFT algebras. In Section 3 we state the main theorem of this paper on solutions to refined quantum master equation and study various consequences, as summarized above. In Section 4, we study the notion of QFT integral and derive Frobenius manifold structure in the semi-classical case. The last section is devoted to a proof of our main theorem. Proofs of some technical propositions used in the main text are in Appendix A and B.

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2. Preliminary

This section is a brief summary of the basic mathematical setting of the previous paper [1]. We shall also fix notations and conventions. The presentation here is in somewhat different order and, perhaps, more streamlined since it is without the attempts to justify whether our scheme is indeed about quantum field theory which complicated the previous document.

A notable addition in this paper is that we have isolated the definition of a QFT complex from a BV QFT algebra as a core structure. We prove two important propositions on QFT complex. General (both commutative and non-commutative) QFT algebra is a QFT complex with certain compatible additional algebraic structure, where the compatibility shall be always stated as suitable \hbar -divisibility conditions.

In the remaining part of this paper we shall use the Einstein summation convention that the repeated upper and lower indices are summed over without the summation notation. It is also understood that an equality labeled by unrepeated indices shall mean that it is valid for all the ranges of the indices, unless otherwise specified.²

Fix a ground field \mathbb{k} of characteristic zero, usually \mathbb{R} or \mathbb{C} . Let \mathcal{C} be a \mathbb{Z} -graded \mathbb{k} -vector space and set

$$\mathcal{C}[[\hbar]] = \left\{ \sum_{n \geq 0} \hbar^n a^{(n)} \mid a^{(n)} \in \mathcal{C} \right\},$$

² For example, $A_{\alpha\beta}{}^\rho A_{\rho\gamma}{}^\sigma = A_{\beta\rho}{}^\rho A_{\alpha\rho}{}^\sigma$ means that $\sum_\rho A_{\alpha\beta}{}^\rho A_{\rho\gamma}{}^\sigma = \sum_\rho A_{\beta\rho}{}^\rho A_{\alpha\rho}{}^\sigma$ for all $\alpha, \beta, \gamma, \sigma$.

where \hbar is a formal parameter (the formal Planck constant). The \mathbb{Z} -grading is specified by ghost number and $|\mathbf{a}|$ denotes the ghost number of \mathbf{a} . We set $|\hbar| = 0$. We shall denote an element of $\mathcal{C}[[\hbar]]$ by an upright **bold** letter, i.e., $\mathbf{a} \in \mathcal{C}[[\hbar]]$, and an element of \mathcal{C} by an *italic* letter, i.e., $a \in \mathcal{C}$, such that formal power series expansion of an element $\mathbf{a} \in \mathcal{C}[[\hbar]]$ shall be denoted as $\mathbf{a} = a^{(0)} + \hbar a^{(1)} + \hbar^2 a^{(2)} + \dots$, where $a^{(n)} \in \mathcal{C}$ for all $n = 0, 1, 2, \dots$. We shall often denote $a^{(0)}$ by a . Projection of any structure parametrized by \hbar from $\mathcal{C}[[\hbar]]$ to \mathcal{C} will be called taking classical limit. In general, anything in formal power series of \hbar is denoted by an upright **bold** letter, such as \mathbf{F} , while a slanted **bold** letter, such as \mathbf{T} , denote something in formal Laurent series of \hbar .

2.1. QFT Complex

Consider a sequence $\mathbf{K} = K^{(0)} + \hbar K^{(1)} + \hbar^2 K^{(2)} + \dots$ of \mathbb{k} -linear maps on \mathcal{C} into \mathcal{C} parametrized by \hbar with ghost number 1. The action of \mathbf{K} on $\mathcal{C}[[\hbar]]$ is defined by the $\mathbb{k}[[\hbar]]$ -linearity and \hbar -adic continuity:

$$\mathbf{K}\mathbf{a} := \sum_{n=0}^{\infty} \sum_{j=0}^n K^{(j)} a^{(n-j)}.$$

We shall usually denote $K^{(0)}$ by Q .

Definition 2.1. The pair $(\mathcal{C}[[\hbar]], \mathbf{K})$ defines the structure of a QFT complex on \mathcal{C} if $\mathbf{K}^2 = 0$.

The structure of QFT complex on \mathcal{C} is considered modulo the natural automorphism on $\mathcal{C}[[\hbar]]$, which is a sequence $\mathbf{g} = 1 + \hbar g^{(1)} + \hbar^2 g^{(2)} + \dots$ of \mathbb{k} -linear maps on \mathcal{C} into itself parametrized by \hbar with ghost number 0 such that $\mathbf{g}|_{\hbar=0} = 1$. The action of \mathbf{g} on $\mathcal{C}[[\hbar]]$ is defined the $\mathbb{k}[[\hbar]]$ -linearity and \hbar -adic continuity. Such an automorphism send \mathbf{K} to $\mathbf{K}' = \mathbf{g}\mathbf{K}\mathbf{g}^{-1}$:

$$\mathbf{K} = Q + \hbar K^{(1)} + \dots \quad \rightarrow \quad \mathbf{K}' = Q + \hbar (K^{(1)} + g^{(1)}Q - Qg^{(1)}) + \dots.$$

The leading relations for the condition $\mathbf{K}^2 = 0$ are

$$\begin{aligned} Q^2 &= 0, \\ QK^{(1)} + K^{(1)}Q &= 0, \\ K^{(1)}K^{(1)} + QK^{(2)} + K^{(2)}Q &= 0, \end{aligned}$$

etc. Thus (\mathcal{C}, Q) is a cochain complex over \mathbb{k} called the underlying classical cochain complex (in the QFT complex). Note that the structure (\mathcal{C}, Q) is invariant under the natural automorphisms on $\mathcal{C}[[\hbar]]$.

Definition 2.2. *The cohomology of a QFT complex $(\mathcal{C}[[\hbar]], \mathbf{K})$ is the cohomology H of the underlying classical cochain complex (\mathcal{C}, Q) , which is fixed by the automorphism on $\mathcal{C}[[\hbar]]$.*

A morphism of QFT complex from $(V[[\hbar]], \mathbf{K}_V)$ to $(W[[\hbar]], \mathbf{K}_W)$ is a sequence $\mathbf{f}_{VW} = f_{VW}^{(0)} + \hbar f_{VW}^{(1)} + \dots$ of \mathbb{k} -linear maps on V to W parametrized by \hbar of ghost number 0, such that $\mathbf{f}_{VW} \mathbf{K}_V = \mathbf{K}_W \mathbf{f}_{VW}$. We shall often denote $f_{VW}^{(0)}$ by f_{VW} . The leading relations are

$$\begin{aligned} f_{VW} Q_V &= Q_W f_{VW}, \\ f_{VW} K_V^{(1)} + f_{VW}^{(1)} Q_V &= K_W^{(1)} f_{VW} + Q_W f_{VW}^{(1)}. \end{aligned}$$

Thus, in particular, f_{VW} is a cochain map of the underlying classical cochain complex.

Definition 2.3. *A quasi-isomorphism of QFT complexes is a morphism of QFT complexes whose underlying classical cochain map induces an isomorphism on the cohomology.*

A quantum homotopy is a sequence

$$\mathbf{s}_{VW} = s_{VW} + \hbar s_{VW}^{(1)} + \hbar^2 s_{VW}^{(2)} + \dots$$

of \mathbb{k} -linear maps on V into W with ghost number -1 parametrized by \hbar . We shall often denote $s_{VW}^{(0)}$ by s_{VW} . Morphisms of QFT complex \mathbf{f}_{VW} and \mathbf{f}'_{VW} are quantum homotopic if

$$\mathbf{f}'_{VW} - \mathbf{f}_{VW} = \mathbf{s}_{VW} \mathbf{K}_V + \mathbf{K}_W \mathbf{s}_{VW}$$

for some quantum homotopy \mathbf{s}_{VW} . The leading relations are

$$\begin{aligned} f'_{VW} - f_{VW} &= s_{VW} K_V + Q_W s_{VW}, \\ f'_{VW} - f_{VW}^{(1)} &= s_{VW} K_V^{(1)} + K_W^{(1)} s_{VW} + s_{VW}^{(1)} Q_V + Q_W s_{VW}^{(1)}. \end{aligned}$$

Thus, in particular, f'_{VW} and f_{VW} are cochain homotopic. Both quantum morphism and quantum homotopy are defined modulo automorphisms \mathbf{g}_{VV} and \mathbf{g}_{WW} such as $\mathbf{f}_{VW} \rightarrow \mathbf{g}_{WW} \mathbf{f}_{VW} \mathbf{g}_{VV}^{-1}$.

Definition 2.4. *A QFT complex $(\mathcal{C}[[\hbar]], \mathbf{K})$ is called on-shell if $\mathbf{K}|_{\hbar=0} = 0$.*

Remark 2.1. The prefix “on-shell” is motivated by an analogy with the classical equation of motion, which corresponds to the vanishing loci of Q regarded as an odd vector field over the space of classical fields.

The following is the theorem in section 3 of the previous paper [1]:

Theorem 2.1. *On cohomology H of a QFT complex $(\mathcal{C}[[\hbar]], \mathbf{K} = Q + \hbar K^{(1)} + \dots)$ there is the structure $(H[[\hbar]], \mathbf{\kappa} = \hbar \kappa^{(1)} + \hbar^2 \kappa^{(2)} + \dots)$ of an on-shell QFT complex. Also, a quasi-isomorphism $f : (H, 0) \rightarrow (\mathcal{C}, Q)$ of the classical complex which induces the identity map on H has an extension to a morphism $\mathbf{f} = f + \hbar f^{(1)} + \hbar^2 f^{(2)} + \dots$ of QFT complexes, $\mathbf{K}\mathbf{f} = \mathbf{f}\mathbf{\kappa}$, which is defined uniquely up to quantum homotopy and automorphisms.*

The leading conditions for $\mathbf{K}\mathbf{f} = \mathbf{f}\mathbf{\kappa}$ are

$$\begin{aligned} Qf &= 0, \\ K^{(1)}f + Qf^{(1)} &= f\kappa^{(1)}. \end{aligned}$$

Let $\mathbf{\kappa} = \hbar^n \kappa^{(n)} + \hbar^{n+1} \kappa^{(n+1)} + \dots$ such that $\kappa^{(n)} \neq 0$. Then $\kappa^{(n)}$ remains invariant under automorphism on $H[[\hbar]]$. The quasi-isomorphism $f : (H, 0) \rightarrow (\mathcal{C}, Q)$ in the above proposition is nothing but a \mathbb{k} -linear way of choosing representative of each and every cohomology class. Then $Qf(x) = 0$ for any $x \in H$ and the Q -cohomology class of $f(x)$ is x , i.e., $[f(x)] = x$, so that f induces the identity map on H . Theorem 2.1 implies that the structures $(H[[\hbar]], \mathbf{\kappa})$ and $(\mathcal{C}[[\hbar]], \mathbf{K})$ on H and \mathcal{C} , respectively, are quasi-isomorphic as QFT complex via \mathbf{f} .

Remark 2.2. Note that the both $(H[[\hbar]], \mathbf{\kappa})$ and $(\mathcal{C}[[\hbar]], \mathbf{K})$ are also cochain complexes over the formal power series ring $\mathbb{k}[[\hbar]]$ and \mathbf{f} as a $\mathbb{k}[[\hbar]]$ -linear map on $H[[\hbar]]$ into $\mathcal{C}[[\hbar]]$ is a cochain map since $\mathbf{K}\mathbf{f} = \mathbf{f}\mathbf{\kappa}$. Recall that a cochain map induces a morphism on the cohomology. The cohomology of a cochain complex $(V[[\hbar]], \mathbf{K}_V)$ over $\mathbb{k}[[\hbar]]$ is, of course, the quotient $\mathbb{k}[[\hbar]]$ -module $\text{Ker } \mathbf{K}_V / \text{Im } \mathbf{K}_V$. A cochain map is a quasi-isomorphism (of cochain complex) if it induces an isomorphism on the cohomology. Hence, our definitions of the cohomology of QFT complex and quasi-isomorphism of QFT complex could be vexing. The following two propositions shall be clarifying, which proofs are located in Appendix A for the sake of streamlined presentation.

Proposition 2.1. *Any $\boldsymbol{\eta} \in \mathcal{C}[[\hbar]]^{|\boldsymbol{\eta}|}$ satisfying*

$$\mathbf{K}\boldsymbol{\eta} = 0,$$

can be expressed as

$$\boldsymbol{\eta} = \mathbf{f}(\mathbf{x}) + \mathbf{K}\boldsymbol{\lambda},$$

for certain pair $(\mathbf{x}, \boldsymbol{\lambda}) \in H[[\hbar]]^{|\boldsymbol{\eta}|} \oplus \mathcal{C}[[\hbar]]^{|\boldsymbol{\eta}|-1}$ such that

$$\boldsymbol{\kappa}\mathbf{x} = 0.$$

Let $(\mathbf{x}', \boldsymbol{\lambda}') \in H[[\hbar]]^{|\boldsymbol{\eta}|} \oplus \mathcal{C}[[\hbar]]^{|\boldsymbol{\eta}|-1}$ be any other pair satisfying $\boldsymbol{\eta} = \mathbf{f}(\mathbf{x}') + \mathbf{K}\boldsymbol{\lambda}'$. Then there is certain pair $(\mathbf{y}, \boldsymbol{\zeta}) \in H[[\hbar]]^{|\boldsymbol{\eta}|-1} \oplus \mathcal{C}[[\hbar]]^{|\boldsymbol{\eta}|-2}$ such that

$$\begin{aligned} \mathbf{x}' - \mathbf{x} &= \boldsymbol{\kappa}\mathbf{y}, \\ \boldsymbol{\lambda}' - \boldsymbol{\lambda} &= \mathbf{f}(\mathbf{y}) + \mathbf{K}\boldsymbol{\zeta}, \end{aligned}$$

Proposition 2.2. A pair $\{\mathbf{x}, \boldsymbol{\lambda}\} \in H[[\hbar]]^{|\mathbf{x}|} \oplus \mathcal{C}[[\hbar]]^{|\mathbf{x}|-1}$ satisfies

$$\mathbf{f}(\mathbf{x}) = \mathbf{K}\boldsymbol{\lambda}$$

if and only if there is a pair $\{\mathbf{y}, \boldsymbol{\zeta}\} \in H[[\hbar]]^{|\mathbf{x}|-1} \oplus \mathcal{C}[[\hbar]]^{|\mathbf{x}|-2}$ such that

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\kappa}\mathbf{y}, \\ \boldsymbol{\lambda} &= \mathbf{f}(\mathbf{y}) + \mathbf{K}\boldsymbol{\zeta}. \end{aligned}$$

Combining the above two propositions, we have

Lemma 2.1. The structures $(H[[\hbar]], \boldsymbol{\kappa})$ and $(\mathcal{C}[[\hbar]], \mathbf{K})$ of QFT complex on H and \mathcal{C} , respectively, are quasi-isomorphic (via \mathbf{f}) also as cochain complex over $\mathbb{k}[[\hbar]]$.

Remark 2.3. Why didn't we define cohomology of a QFT complex $(\mathcal{C}[[\hbar]], \mathbf{K})$, which is automatically a cochain complex over $\mathbb{k}[[\hbar]]$, by the standard one? There are at least three reasons not to do that. One reason is that the natural automorphism on $\mathcal{C}[[\hbar]]$ does not fix \mathbf{K} so that \mathbf{K} -cohomology may not have intrinsic meaning for our purpose. Another reason is that we can hardly expect the \mathbf{K} -cohomology group is free $\mathbb{k}[[\hbar]]$ -module in general, while both $H[[\hbar]]$ and $\mathcal{C}[[\hbar]]$ are free. The final and perhaps the most important reason for us is that we are losing some crucial information, encoded by $\boldsymbol{\kappa}$, by taking \mathbf{K} -cohomology. It may be plausible that the totality of free resolutions of the \mathbf{K} -cohomology module may recover the lost information, while the on-shell QFT complex $(\mathcal{C}[[\hbar]], \boldsymbol{\kappa})$ could be "the best" model.

The purposes of propositions 2.1 and 2.2 are for more than lemma 2.1. Consider, for example, proposition 2.1. Note that its classical limit is completely standard. Let $\eta \in \mathcal{C}^{|\eta|}$ satisfying $Q\eta = 0$. Then there is certain pair $(x, \lambda) \in H^{|\eta|} \oplus \mathcal{C}^{|\eta|-1}$ such that

$$\eta = f(x) + Q\lambda.$$

Let $(x', \lambda') \in H^{|\eta|} \oplus \mathcal{C}^{|\eta|-1}$ is any other pair satisfying $\eta = f(x') + Q\lambda'$ then there is a pair $(y, \zeta) \in H^{|\eta|-1} \oplus \mathcal{C}^{|\eta|-2}$ such that

$$\begin{aligned} x' - x &= 0, \\ \lambda' - \lambda &= f(y) + Q\zeta, \end{aligned}$$

since $\kappa|_{\hbar=0} = 0$. The above statements mean, simply, that $x = [\eta] = [f(x)]$ (uniqueness), while the space of ambiguities of λ is $\text{Ker } Q$, which has a decomposition $\text{Im } f \oplus \text{Im } Q$. And there is no "correlation" between the (zero) ambiguity of x and the ambiguity of λ . The situation is rather different in general. The ambiguities of \mathbf{x} and $\boldsymbol{\lambda}$ are correlated and the space of ambiguities of $\boldsymbol{\lambda}$ is not necessarily $\text{Ker } \mathbf{K}$, since $\mathbf{K}(\boldsymbol{\lambda}' - \boldsymbol{\lambda}) = \mathbf{f}(\boldsymbol{\kappa}\mathbf{y}) \neq 0$ in general. Such correlation vanishes if $\boldsymbol{\kappa} = 0$.

Corollary 2.1. *Let $\boldsymbol{\kappa} = 0$. Then any $\boldsymbol{\eta} \in \mathcal{C}[[\hbar]]^{|\eta|}$ satisfying $\mathbf{K}\boldsymbol{\eta} = 0$ can be written as $\boldsymbol{\eta} = \mathbf{f}(\mathbf{x}) + \mathbf{K}\boldsymbol{\zeta}$ for unique $\mathbf{x} \in H[[\hbar]]$ and some $\boldsymbol{\zeta} \in \mathcal{C}[[\hbar]]$, which is defined modulo $\text{Ker } \mathbf{K}$.*

Proof. Proposition 2.1 with $\boldsymbol{\kappa} = 0$. \square

Corollary 2.2. *Let $\boldsymbol{\kappa} = 0$ and let $\mathbf{f}(\mathbf{x}) = \mathbf{K}\boldsymbol{\lambda}$. Then $\mathbf{x} = 0$ and $\mathbf{K}\boldsymbol{\lambda} = 0$.*

Proof. Proposition 2.2 with $\boldsymbol{\kappa} = 0$. \square

Finally we state a simple proposition, which deserve some attention.

Proposition 2.3. *Let $\boldsymbol{\kappa} = 0$. Assume that we have the following type of equality*

$$\hbar\boldsymbol{\eta} = \mathbf{K}\boldsymbol{\lambda},$$

where $\boldsymbol{\eta} \in \mathcal{C}[[\hbar]]^{|\eta|}$ and $\boldsymbol{\lambda} \in \mathcal{C}[[\hbar]]^{|\eta|-1}$. Then there exist some $\boldsymbol{\xi} \in \mathcal{C}[[\hbar]]$ such that $\boldsymbol{\eta} = \mathbf{K}\boldsymbol{\xi}$.

Proof. From the classical limit we have $Q\lambda = 0$. Hence, $\lambda = f([\lambda]) + Q\rho$ for some $\rho \in \mathcal{C}$. It follows that the expression $\lambda - \mathbf{f}([\lambda]) - \mathbf{K}\rho$ is divisible by \hbar . Define

$$\hbar\xi := \lambda - \mathbf{f}([\lambda]) - \mathbf{K}\rho$$

so that $\xi \in \mathcal{C}[[\hbar]]^{|M|-1}$. Applying \mathbf{K} to the above we obtain that $\hbar\mathbf{K}\xi = \mathbf{K}\lambda$. Hence we have $\hbar\mathbf{M} = \hbar\mathbf{K}\xi$, which is equivalent to $\mathbf{M} = \mathbf{K}\xi$. \square

Remark 2.4. We should emphasize that the condition $\hbar\eta = \mathbf{K}\lambda$ does not necessarily implies that $\eta = \mathbf{K}\xi$ for some $\xi \in \mathcal{C}[[\hbar]]$, since the obvious candidate $\xi = \frac{1}{\hbar}\lambda$ may not belong to $\mathcal{C}[[\hbar]]$ - λ may not be divisible by \hbar in $\mathcal{C}[[\hbar]]$. The notion of QFT complex and the condition divisibility by \hbar shall plays prominent roles in developing our theory.

2.2. Observable, QFT Cycle and Quantum Expectation Value

Definition 2.5. An observable x is an element of H such that $\kappa x = 0$, i.e., $\kappa^{(\ell)}x = 0$ for $\forall \ell = 1, 2, 3, \dots$.

Let $x \in H$ be an observable. Then $Qf(x) = 0$, where $f(x) \in \mathcal{C}^{|x|}$ and the Q -cohomology class $[f(x)]$ of $f(x)$ is x . We say $f(x)$ a *classical representative* of the observable x . Then theorem 2.1 means that $\mathbf{f}(x) = f(x) + \hbar f^{(1)}(x) + \dots \in \mathcal{C}[[\hbar]]^{|x|}$ satisfies $\mathbf{K}\mathbf{f}(x) = 0$. We say $\mathbf{f}(x)$ a *quantum representative* of the observable x . Now consider an element $y \in H$ such that $\kappa y \neq 0$. We still have $Qf(y) = 0$, while theorem 2.1 means that \hbar -correction to $f(y)$ such that it is annihilated by \mathbf{K} does not exist. We say QFT complex is *anomaly-free* if $\kappa = 0$ on its cohomology. We call an element $y \in H$ with $\kappa y \neq 0$ an invisible. Note that observable and invisible are indistinguishable in the classical limit. The existence of invisibles shall be identified with that of fundamental quantum symmetry in a sequel.

Definition 2.6. A QFT cycle of dimension N for a QFT complex $(\mathcal{C}[[\hbar]], \mathbf{K})$ is a sequence $\mathbf{c} = \mathbf{c}^{(0)} + \hbar\mathbf{c}^{(1)} + \hbar^2\mathbf{c}^{(2)} + \dots$ of \mathbb{k} -linear maps $c^{(\ell)}$, $\ell = 0, 1, 2, \dots$, on \mathcal{C} into \mathbb{k} of ghost number $-N$ parametrized by \hbar satisfying

$$\mathbf{c}\mathbf{K} = 0.$$

Two QFT cycles \mathbf{c}, \mathbf{c}' of dimension N are quantum homotopic if there is a sequence $\mathbf{r} = r^{(0)} + \hbar r^{(1)} + \hbar^2 r^{(2)} + \dots$ of \mathbb{k} -linear maps $r^{(\ell)}$, $\ell = 0, 1, 2, \dots$, on \mathcal{C} into \mathbb{k} of ghost number $-N - 1$ parametrized by \hbar such that $\mathbf{c}' - \mathbf{c} = \mathbf{r}\mathbf{K}$.

Remark 2.5. Note that the ghost number of \mathbb{k} and $\mathbb{k}[[\hbar]]$ is concentrated zero. Hence the sequence maps $\mathfrak{c}^{(0)}, \mathfrak{c}^{(1)}, \dots$ should be zero maps on \mathcal{C}^j for $j \neq N$.

We recall that theorem 2.1 gives a sequence $\mathbf{f} := f + \hbar f^{(1)} + \hbar^2 f^{(2)} + \dots$ of \mathbb{k} -linear maps parametrized by \hbar on H into \mathcal{C} defined up to quantum homotopy satisfying $\mathbf{K}\mathbf{f} = \mathbf{f}\mathbf{K}$. We can compose the map \mathbf{f} , regarded as a $\mathbb{k}[[\hbar]]$ -linear map on $H[[\hbar]]$ into $\mathcal{C}[[\hbar]]$, with the map $\mathbf{c} := \mathfrak{c}^{(0)} + \hbar \mathfrak{c}^{(1)} + \hbar^2 \mathfrak{c}^{(2)} + \dots$, regarded as a $\mathbb{k}[[\hbar]]$ -linear map on $\mathcal{C}[[\hbar]]$ into $\mathbb{k}[[\hbar]]$, to obtain a sequence³

$$\mathfrak{t} := \chi \mathbf{f} = \iota^{(0)} + \hbar \iota^{(1)} + \hbar^2 \iota^{(2)} + \dots$$

of \mathbb{k} -linear maps parametrized by \hbar on H into \mathbb{k} such that $\iota^{(n)} = \sum_{\ell=0}^n \mathfrak{c}^{(n-\ell)} f^{(\ell)}$. The ambiguity of \mathfrak{t} due to the ambiguities of \mathbf{f} and \mathbf{c} up to quantum homotopy, $\mathbf{f} \sim \mathbf{f}' = \mathbf{f} + \mathbf{K}\mathbf{s} + \mathbf{s}\mathbf{K}$ and $\mathbf{c} \sim \mathbf{c}' = \mathbf{c} + \mathbf{r}\mathbf{K}$, is $\mathfrak{t}' - \mathfrak{t} \equiv \mathbf{c}'\mathbf{f}' - \mathbf{c}\mathbf{f} = (\mathbf{c}\mathbf{s} + \mathbf{r}\mathbf{f} + \mathbf{r}\mathbf{K}\mathbf{s})\mathbf{K}$.

Definition 2.7. Let $x \in H$ be an observable. The quantum expectation value of the observable x is $\mathfrak{t}(x) := \mathbf{c}\mathbf{f}(x)$, which is a quantum homotopy invariant.

2.3. BV QFT Algebra and BV QFT

Now we turn to BV QFT algebra and BV QFT, which notions were motivated by the celebrated Batalin-Vilkovisky quantization scheme [13]. Let (\mathcal{C}, \cdot) be a \mathbb{Z} -graded \mathbb{k} -vector space with a bilinear product \cdot of ghost number zero. Then there is a canonical $\mathbb{k}[[\hbar]]$ -bilinear product on $\mathcal{C}[[\hbar]]$, denoted by the same notation \cdot , induced from \mathcal{C} by \hbar -adic continuity, i.e.,

$$\mathbf{a} \cdot \mathbf{b} = a^{(0)} \cdot b^{(0)} + \sum_{n=1}^{\infty} \hbar^n \sum_{i+j=n} a^{(i)} \cdot b^{(j)}.$$

Definition 2.8. A BV QFT algebra (with unit 1) is a triple $(\mathcal{C}[[\hbar]], \mathbf{K}, \cdot)$, where the pair $(\mathcal{C}[[\hbar]], \mathbf{K})$ is QFT complex and the bilinear product \cdot is graded-commutative and associative such that

- quantum unit: $\mathbf{K}1 = 0$ and $1 \cdot \mathbf{a} = \mathbf{a}$ for all $\mathbf{a} \in \mathcal{C}[[\hbar]]$.
- \hbar -condition: the failure of \mathbf{K} being a derivation of the product \cdot is divisible by \hbar and the binary operation measuring the failure is a derivation of the product.

³ The composition symbol is omitted throughout this paper.

From the \hbar -condition, it is convenient to introduce a $\mathbb{k}[[\hbar]]$ -bilinear map

$$(\cdot, \cdot)_{\hbar} : \mathcal{C}[[\hbar]]^{k_1} \otimes \mathcal{C}[[\hbar]]^{k_2} \longrightarrow \mathcal{C}[[\hbar]]^{k_1+k_2+1}$$

called BV bracket by the formula

$$-\hbar(-1)^{|\mathbf{a}|}(\mathbf{a}, \mathbf{b})_{\hbar} := \mathbf{K}(\mathbf{a} \cdot \mathbf{b}) - \mathbf{K}\mathbf{a} \cdot \mathbf{b} - (-1)^{|\mathbf{a}|}\mathbf{a} \cdot \mathbf{K}\mathbf{b}. \quad (2.1)$$

Then, by definition, the bracket is a (graded) derivation of the product (Poisson-law)

$$(\mathbf{a}, \mathbf{b} \cdot \mathbf{c})_{\hbar} = (\mathbf{a}, \mathbf{b})_{\hbar} \cdot \mathbf{c} + (-1)^{(|\mathbf{a}|+1)|\mathbf{b}|}\mathbf{b} \cdot (\mathbf{a}, \mathbf{c})_{\hbar}, \quad (2.2)$$

and the unit 1 is in its center. It follows that the bracket satisfies the graded commutativity and the graded Jacobi-identity such that \mathbf{K} is a derivation of it:

$$\begin{aligned} (\mathbf{a}, \mathbf{b})_{\hbar} &= -(-1)^{(|\mathbf{a}|+1)(|\mathbf{b}|+1)}(\mathbf{b}, \mathbf{a})_{\hbar}, \\ (\mathbf{a}, (\mathbf{b}, \mathbf{c})_{\hbar})_{\hbar} &= ((\mathbf{a}, \mathbf{b}), \mathbf{c})_{\hbar} + (-1)^{(|\mathbf{a}|+1)(|\mathbf{b}|+1)}(\mathbf{b}, (\mathbf{a}, \mathbf{c})_{\hbar})_{\hbar}, \\ \mathbf{K}(\mathbf{a}, \mathbf{b})_{\hbar} &= (\mathbf{K}\mathbf{a}, \mathbf{b})_{\hbar} + (-1)^{|\mathbf{a}|+1}(\mathbf{a}, \mathbf{K}\mathbf{b})_{\hbar}. \end{aligned} \quad (2.3)$$

We call the triple

$$(\mathcal{C}[[\hbar]], \mathbf{K}, (\cdot, \cdot)_{\hbar})$$

the quantum descendant algebra (of the the BV QFT algebra).

Remark 2.6. Note that $(\cdot, \cdot)_{\hbar} = (\cdot, \cdot) + \hbar(\cdot, \cdot)^{(1)} + \hbar^2(\cdot, \cdot)^{(2)} + \dots$ is a sequence of \mathbb{k} -bilinear maps, parametrized by \hbar , on $\mathcal{C} \otimes \mathcal{C}$ into \mathcal{C} . Strictly speaking the traditional BV bracket does not depends on \hbar . In the previous paper we have abused notation by not distinguishing $(\cdot, \cdot)_{\hbar}$ with its classical limit (\cdot, \cdot) .

Remark 2.7. The \hbar -condition in the definition of BV QFT algebra can be relaxed such that the failure of the binary operation being a derivation of the product is divisible by \hbar^2 , the failure of the resulting ternary operation being a derivation of the product is divisible by \hbar^3 , to be repeated ad infinitum. Such QFT algebra may be called (graded)-commutative binary QFT algebra and a similar story as in this paper can be developed. It is just another (not-general) example of QFT algebra, which may be studied elsewhere. For us two of the essential properties of general QFT algebra are that the underlying QFT complex and its interplay with additional algebraic structure with the compatibility stated in term of sequences of divisibility condition by \hbar .

By definition the classical limit Q of \mathbf{K} satisfies $Q^2 = 0$ and is a derivation of the product, i.e., $Q(a \cdot b) = Qa \cdot b + (-1)^{|a|}a \cdot Qb$. Also $Q1 = 0$. Thus the classical limit (\mathcal{C}, Q, \cdot) of a BV QFT algebra is a differential graded commutative and associative algebra (CDGA)

over \mathbb{k} with unit 1. It also follows that the classical limit $(\mathcal{C}, Q, (\cdot, \cdot))$ of the descendant algebra is a DG0LA over \mathbb{k} .⁴ Hence the quadruple $(\mathcal{C}, Q, \cdot, (\cdot, \cdot))$ is a differential 0-algebra since the bracket is a derivation of the product. The cohomology H of (\mathcal{C}, Q) is also the cohomology of the CDGA and the DG0LA as well as the BV QFT algebra. BV QFT algebra is also defined up to natural automorphism on $\mathcal{C}[[\hbar]]$, which fix all the classical limits.

Observables of a BV QFT algebra are the observables in its QFT complex, elements in H annihilated by $\kappa = \hbar\kappa^{(1)} + \hbar^2\kappa^{(2)} + \dots$. From the quantum unity $\mathbf{K}1 = Q1 = 0$, in the definition of BV QFT algebra, there is a distinguished element $e \in H^0$ corresponding to the cohomology class [1] of the unit 1 such that $\kappa e = 0$. It is natural to fix f and its extension \mathbf{f} to a morphism of QFT complex such that $f(e) = \mathbf{f}(e) = 1$, the unit in \mathcal{C} .

Definition 2.9. *A BV QFT with ghost number anomaly $N \in \mathbb{Z}$ is a BV QFT algebra (with unit 1) together with a QFT cycle $\mathbf{c} = \mathbf{c}^{(0)} + \hbar\mathbf{c}^{(1)} + \dots$ of dimension N .*

We call the quantum expectation value $\mathbf{cf}(e) = \mathbf{c}(1) \in \mathbb{k}[[\hbar]]$ of $e \in H^0$ the partition function \mathbf{Z} of BV QFT. It follows that the partition function vanishes unless $N = 0$. We may consider a class of BV QFT with vanishing ghost number anomaly such that $\mathbf{c}(1)|_{\hbar=0} \equiv \mathbf{c}^{(0)}(1) \neq 0$. Then the quantum expectation values can be normalized by multiplying \mathbf{Z}^{-1} . Such a theory is called unital BV QFT, with the normalization being understood.

Definition 2.10. *An unital BV QFT is a BV QFT algebra (with unit 1) together with a QFT cycle $\mathbf{c} = \mathbf{c}^{(0)} + \hbar\mathbf{c}^{(1)} + \dots$ with dimension $N = 0$ such that $\mathbf{c}(1) = 1 \in \mathbb{k}$.*

From now on we are going to adopt the time honored notation for expectation value such that, for $\forall x \in H$,

$$\langle \mathbf{f}(x) \rangle = \mathbf{cf}(x),$$

where $\mathbf{f}(x) \in \mathcal{C}[[\hbar]]^{|x|}$. We emphasize that the expectation value $\langle \mathbf{f}(x) \rangle$ depends only on the quantum homotopy class of the QFT cycle \mathbf{c} if and only if $\kappa(x) = 0$. We also emphasize that the condition $\kappa(x) = 0$ implies that $\mathbf{Kf}(x) = 0$ and the expectation value $\langle \mathbf{f}(a) \rangle$ depends only on the quantum homotopy class of \mathbf{f} .

Remark 2.8. It is perhaps useful to tweak the notation as follows. Let $O \in \mathcal{C}$ be such that $QO = 0$, then $O = f([O])$ up to homotopy $Q\Lambda$ for some $\Lambda \in \mathcal{C}$. Assume that $\kappa[O] =$

⁴ In our convention, a DG0LA $(\mathcal{C}, Q, (\cdot, \cdot))$ is a differential graded algebra after shifting the ghost number by -1 .

0. Set $\mathbf{O} = \mathbf{f}([O]) \in \mathcal{C}[[\hbar]]$ up to quantum homotopy $\mathbf{K}\Lambda$ for some $\Lambda \in \mathcal{C}[[\hbar]]$. Then $\mathbf{K}\mathbf{O} = 0$ and

$$\langle \mathbf{O} \rangle = \langle \mathbf{O} + \mathbf{K}\Lambda \rangle,$$

since $\langle \mathbf{K}\Lambda \rangle = \mathbf{c}\mathbf{K}\Lambda = 0$. The variation of the QFT cycle \mathbf{c} preserving its quantum homotopy class correspond to homologous deformation of Lagrangian subspace on where the BV-Feynman path integral is defined in the BV quantization scheme.

3. Statement of the Main Theorem and Its Consequences

Fix a BV QFT algebra (with unit 1) $(\mathcal{C}[[\hbar]], \mathbf{K}, \cdot)$ with quantum descendant algebra $(\mathcal{C}[[\hbar]], \mathbf{K}, (\cdot, \cdot)_\hbar)$ and let (\mathcal{C}, Q, \cdot) and $(\mathcal{C}, Q, (\cdot, \cdot))$, respectively, be their classical limits. In this paper we consider the case that (i) the cohomology H is finite-dimensional for each ghost number and (ii) $\mathbf{k} = 0$ on H .

From the assumption (i), it is convenient to introduce a homogeneous basis $\{e_\alpha\}$ of H . Define a \mathbb{k} -linear map $f : H \rightarrow \mathcal{C}$ of ghost number 0 by choosing a representative O_α of each $e_\alpha \in H$ such that

$$f(e_\alpha) = O_\alpha \in \mathcal{C}^{|e_\alpha|},$$

i.e., $QO_\alpha = 0$ and the Q -cohomology class $[O_\alpha]$ of O_α is e_α . Then $Qf = 0$. It is convenient to fix a basis $\{e_\alpha\}$ of H such that one of its component, say e_0 , is the distinguished element $e \in H^0$, i.e.,

$$f(e_0) = O_0 = 1.$$

It follows that f is a quasi-isomorphism of cochain complexes $f : (H, 0) \rightarrow (\mathcal{C}, Q)$, which induces the identity map on H . The graded-commutative and associative product $m_2 : H \otimes H \rightarrow H$ of ghost number 0 on H can be specified structure constants $m_{\alpha\beta}^\gamma \in \mathbb{k}$:

$$m_2(e_\alpha, e_\beta) = m_{\alpha\beta}^\gamma e_\gamma.$$

satisfying

- graded symmetry: $m_{\alpha\beta}^\gamma = (-1)^{|\alpha||\beta|} m_{\beta\alpha}^\gamma$,
- associativity: $m_{\alpha\beta}^\rho m_{\rho\gamma}^\sigma = m_{\beta\gamma}^\rho m_{\alpha\rho}^\sigma$,
- identity: $m_{0\beta}^\gamma = \delta_\beta^\gamma$.

At the cochain level, we have

$$f(e_\alpha) \cdot f(e_\beta) = f(m_2(e_\alpha, e_\beta)) + K\lambda_2(e_\alpha, e_\beta), \quad (3.1)$$

where $\lambda_2 : H \otimes H \longrightarrow \mathcal{C}$ is a \mathbb{k} -bilinear map of ghost number 1. That is, f is a DGA map $f : (H, 0, m_2) \longrightarrow (\mathcal{C}, Q, \cdot)$ up to homotopy.

Let $t_H = \{t^\alpha\}$ be the dual basis of H^* such that $|t^\alpha| + |e_\alpha| = 0$, which is an affine coordinates system on H with a distinguished coordinate t^0 . We denote $\{\partial_\alpha = \partial / \partial t^\alpha\}$ be the corresponding formal partial derivatives acting on $\mathbb{k}[[t_H]]$ as derivations. To save notation we replace $(-1)^{|e_\alpha|} = (-1)^{-|t_\alpha|} = (-1)^{|O_\alpha|} = (-1)^{|\mathbf{O}_\alpha|}$ by $(-1)^{|\alpha|}$. We shall also use notation $t^{\tilde{\alpha}}$ for $(-1)^{|\alpha|} t^\alpha$ and $\partial_{\tilde{\alpha}} = \partial / \partial t^{\tilde{\alpha}}$.

From the assumption (ii) and theorem 2.1, we have a sequence $\mathbf{f} = f + \hbar f^{(1)} + \dots$ of \mathbb{k} -linear maps on H into \mathcal{C} , parametrized by \hbar , of ghost number zero such that $\mathbf{K}\mathbf{f} = 0$. The image of $e_\alpha \in H$ of the map \mathbf{f} will be denoted by \mathbf{O}_α :

$$\mathbf{f}(e_\alpha) = \mathbf{O}_\alpha = O_\alpha + \hbar O_\alpha^{(1)} + \hbar^2 O_\alpha^{(2)} + \dots \in \mathcal{C}[[\hbar]]^{|e_\alpha|}$$

such that $\mathbf{K}\mathbf{O}_\alpha = 0$ and

$$\mathbf{f}(e_0) = \mathbf{O}_0 = 1.$$

We use the following terminology: O_α is a classical representative of the observable e_α , \mathbf{O}_α is a quantum representative of the observable e_α , O_α is the classical limit $\mathbf{O}_\alpha|_{\hbar=0}$ of \mathbf{O}_α , and \mathbf{O}_α is the quantization of O_α with respect to the quantization map \mathbf{f} .

Theorem 3.1. *There is the structure of $\mathbb{k}[[t_H]]$ -algebra on $H \otimes \mathbb{k}[[t_H]]$ defined by a formal power series 3-tensor $A_{\beta\gamma}{}^\sigma \in \mathbb{k}[[t_H]]$, which satisfies*

– *graded symmetricity:*

$$\begin{aligned} A_{\beta\gamma}{}^\sigma &= (-1)^{|\beta||\gamma|} A_{\gamma\beta}{}^\sigma, \\ \partial_\alpha A_{\beta\gamma}{}^\sigma &= (-1)^{|\alpha||\beta|} \partial_\beta A_{\alpha\gamma}{}^\sigma, \end{aligned}$$

– *associativity:*

$$A_{\alpha\beta}{}^\rho A_{\rho\gamma}{}^\sigma = A_{\beta\gamma}{}^\rho A_{\alpha\rho}{}^\sigma.$$

– *identity:*

$$A_{0\beta}{}^\gamma = \delta_\beta{}^\gamma,$$

– *The homogeneity:*

$$|e_\rho| t^\rho \partial_\rho A_{\alpha\beta}{}^\gamma = (|e_\gamma| - |e_\beta| - |e_\alpha|) A_{\alpha\beta}{}^\gamma.$$

And, there is a distinguished solution Θ of the Maurer-Cartan equation

$$\mathbf{K}\Theta + \frac{1}{2}(\Theta, \Theta)_\hbar = 0,$$

where

$$\Theta = t^\alpha \mathbf{O}_\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} t^{\alpha_n} \dots t^{\alpha_1} \mathbf{O}_{\alpha_1 \dots \alpha_n} \in \mathcal{C}[[t_H, \hbar]]^0,$$

which satisfies

1. quantum master equation:

$$\hbar \partial_\beta \partial_\gamma \Theta = \partial_\beta \Theta \cdot \partial_\gamma \Theta - A_{\beta\gamma}{}^\sigma \partial_\sigma \Theta - \mathbf{K} \Lambda_{\beta\gamma} - (\Theta, \Lambda_{\beta\gamma})_\hbar,$$

for some $\Lambda_{\beta\gamma} \in \mathcal{C}[[t_H, \hbar]]^{|\beta|+|\gamma|-1}$ in quantum gauge.

2. quantum unity: $\partial_0 \Theta = 1$.

Remark 3.1. The quantum gauge condition for $\Lambda_{\alpha\beta}$ shall be stated later.

Remark 3.2. The Maurer-Cartan equation $\mathbf{K}\Theta + \frac{1}{2}(\Theta, \Theta)_\hbar = 0$ is called quantum descendant equation, which is an automatic consequence of the quantum master equation. We remark that the quantum descendant equation is the Batalin-Vilkovisky quantum master equation for a family of QFT.

The quantum master equation together with the unity should be regraded as a system of formal differential equations for $\Theta \in \mathcal{C}[[t_H, \hbar]]^0$, $A_{\alpha\beta}{}^\gamma \in \mathbb{k}[[t_H]]$ and $\Lambda_{\alpha\beta} \in \mathcal{C}[[t_H, \hbar]]^{|\alpha|+|\beta|-1}$ with the initial condition that $\Theta = t^\alpha \mathbf{f}(e_\alpha) \bmod t_H^2$, where $\mathbf{f}(e_\alpha) = \mathbf{O}_\alpha$ and $\mathbf{f}(e_0) = 1$. The theorem claims that the only obstruction to solve the quantum master equation with the unity is $\kappa = \hbar \kappa^{(1)} + \hbar^2 \kappa^{(2)} + \dots$ on H , which vanishes in the present case.

This is illustrated for the 1st order solution in the following example.

Example 3.1. To begin with, consider the quantum master equation modulo t_H :

$$\hbar \mathbf{O}_{\alpha\beta} = \mathbf{O}_\alpha \cdot \mathbf{O}_\beta - m_{\alpha\beta}{}^\gamma \mathbf{O}_\gamma - \mathbf{K} \lambda_{\alpha\beta} \quad (3.2)$$

where $\mathbf{O}_{\alpha\beta} = \partial_\alpha \partial_\beta \Theta \bmod t_H$, $m_{\alpha\beta}{}^\gamma = A_{\alpha\beta}{}^\gamma \bmod t_H$ and $\lambda_{\alpha\beta} = \Lambda_{\alpha\beta} \bmod t_H$ are unknowns. Then we have the following consistency conditions for (3.2):

(1) classical limit:

$$O_\alpha \cdot O_\beta = m_{\alpha\beta}{}^\gamma O_\gamma + Q \lambda_{\alpha\beta},$$

(2) graded commutativity of the product \cdot :

$$(m_{\alpha\beta}{}^\gamma - (-1)^{|\alpha||\beta|} m_{\beta\alpha}{}^\gamma) O_\gamma = -Q (\lambda_{\alpha\beta} - (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha}),$$

(3) unity:

$$O_\beta = m_{0\beta}^\gamma O_\gamma + Q\lambda_{0\beta}, \quad O_\alpha = m_{\alpha 0}^\gamma O_\gamma + Q\lambda_{\alpha 0}.$$

For (1), we note that $O_\alpha \cdot O_\beta \in \text{Ker } Q$ since $QO_\alpha = 0$ and Q is a derivation of the product. Hence $O_\alpha \cdot O_\beta$ can be expressed as $m_{\alpha\beta}^\gamma O_\gamma + Q\lambda_{\alpha\beta}$ for unique $m_{\alpha\beta}^\gamma$ and for some $\lambda_{\alpha\beta} \in \mathcal{C}^{|\alpha|+|\beta|-1}$ modulo $\text{Ker } Q$. From (2) and (3), we conclude that

$$m_{\alpha\beta}^\gamma - (-1)^{|\alpha||\beta|} m_{\beta\alpha}^\gamma = 0, \quad m_{0\beta}^\gamma = \delta_{\beta}^\gamma,$$

as well as $Q(\lambda_{\alpha\beta} - (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha}) = 0$ and $Q\lambda_{0\beta} = Q\lambda_{\alpha 0} = 0$. For $\lambda_{\alpha\beta}$, we may choose them, without loss of generality, to be graded-symmetric, $\lambda_{\alpha\beta} = (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha}$, and satisfy $\lambda_{0\alpha} = 0$. Let $\lambda_{\alpha\beta} = \lambda_{\alpha\beta}$ such that

$$\lambda_{\alpha\beta} = (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha}, \quad \lambda_{0\beta} = 0. \quad (3.3)$$

Then the expression $\mathbf{L}_{\alpha\beta} = \mathbf{O}_\alpha \cdot \mathbf{O}_\beta - m_{\alpha\beta}^\gamma \mathbf{O}_\gamma - \mathbf{K}\lambda_{\alpha\beta}$ is divisible by \hbar and graded symmetric. It also satisfies that $\mathbf{L}_{0\beta} = \mathbf{L}_{\alpha 0} = 0$. Once we define $\mathbf{O}_{\alpha\beta} := \frac{1}{\hbar} \mathbf{L}_{\alpha\beta} \in \mathcal{C}[[\hbar]]^{|\alpha|+|\beta|}$, $\mathbf{O}_{\alpha\beta}$ is graded symmetric $\mathbf{O}_{\alpha\beta} = (-1)^{|\alpha||\beta|} \mathbf{O}_{\beta\alpha}$ and satisfies $\mathbf{O}_{0\beta} = 0$. Hence we have just solved the modulo t_H quantum master equation (3.2) by setting $\Theta = t^\alpha \mathbf{O}_\alpha + \frac{1}{2} t^\beta t^\alpha \mathbf{O}_{\alpha\beta} \bmod t_H^3$, $A_{\alpha\beta}^\gamma = m_{\alpha\beta}^\gamma \bmod t_H$ and $\Lambda_{\alpha\beta} = \lambda_{\alpha\beta} \bmod t_H$ such that the quantum identity modulo t_H^2 is satisfied:

$$\partial_0 \Theta = 1 \bmod t_H^2,$$

and

$$A_{\alpha\beta}^\gamma - (-1)^{|\alpha||\beta|} A_{\beta\alpha}^\gamma = 0 \bmod t_H.$$

For the quantum descent equation, apply \mathbf{K} to the modulo t_H quantum master equation (3.2) to have $\mathbf{K}\mathbf{O}_{\alpha\beta} = -\hbar(-1)^{|\alpha|} (\mathbf{O}_\alpha, \mathbf{O}_\beta)_\hbar$, which implies, together with the initial condition $\mathbf{K}\mathbf{O}_\alpha = 0$, that

$$\mathbf{K}\Theta + \frac{1}{2} (\Theta, \Theta)_\hbar = 0 \bmod t_H^3.$$

Hence we have the quantum descent equation modulo t_H^3 .

The above, perhaps, is not enough as the example for demonstration of our method of solving quantum master equation modulo t_H^n for general $n \geq 2$. Here comes a step by step description of our solution modulo t_H^2 featuring general behavior for higher n . Most of necessary propositions shall be stated as claims, referring to the actual proof in Sect. 4. The following example is useful for pedagogical purpose but some readers may want to skip it in the first reading.

Example 3.2. Now we consider the quantum master equation modulo t_H^2 , which is eq. (3.2) and

$$\hbar \mathbf{O}_{\alpha\beta\gamma} = \mathbf{C}_{\alpha\beta\gamma} - m_{\alpha\beta\gamma}{}^\rho \mathbf{O}_\rho - \mathbf{K} \boldsymbol{\lambda}_{\alpha\beta\gamma}, \quad (3.4)$$

where $\mathbf{O}_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma \boldsymbol{\Theta} \bmod t_H$, $m_{\alpha\beta\gamma}{}^\rho$, and $\boldsymbol{\lambda}_{\alpha\beta\gamma}$ are unknowns, while the expression $\mathbf{C}_{\alpha\beta\gamma}$ is determined by the previous data $\mathbf{O}_\alpha, \mathbf{O}_{\alpha\beta}, m_{\alpha\beta}{}^\gamma, \boldsymbol{\lambda}_{\alpha\beta}$ as follows:

$$\mathbf{C}_{\alpha\beta\gamma} := \mathbf{O}_{\alpha\beta} \cdot \mathbf{O}_\gamma + (-1)^{|\alpha||\beta|} \mathbf{O}_\beta \cdot \mathbf{O}_{\alpha\gamma} - m_{\beta\gamma}{}^\rho \mathbf{O}_{\alpha\rho} - (\mathbf{O}_\alpha, \boldsymbol{\lambda}_{\beta\gamma})_{\hbar}.$$

The consistency of (3.4) in the classical limit requires that the classical limit $C_{\alpha\beta\gamma}$ of $\mathbf{C}_{\alpha\beta\gamma}$ satisfies $QC_{\alpha\beta\gamma} = 0$ (see claim (1) below) such that

$$C_{\alpha\beta\gamma} = m_{\alpha\beta\gamma}{}^\rho O_\rho + Q\lambda_{\alpha\beta\gamma} \quad (3.5)$$

for unique $m_{\alpha\beta\gamma}{}^\rho$ and some $\lambda_{\alpha\beta\gamma} \in \mathcal{C}^{|\alpha|+|\beta|+|\gamma|-1}$ defined modulo $\text{Ker } Q$. Note also that

$$\mathbf{C}_{0\beta\gamma} = \mathbf{C}_{\alpha 0\gamma} = \mathbf{C}_{\alpha\beta 0} = 0, \quad \mathbf{C}_{\alpha\beta\gamma} = (-1)^{|\beta||\gamma|} \mathbf{C}_{\alpha\gamma\beta}.$$

Hence, we have

$$\begin{aligned} m_{0\beta\gamma}{}^\rho O_\rho + Q\lambda_{0\beta\gamma} &= m_{\alpha 0\gamma}{}^\rho O_\rho + Q\lambda_{\alpha 0\gamma} = m_{\alpha\beta 0}{}^\rho O_\rho + Q\lambda_{\alpha\beta 0} = 0, \\ (m_{\alpha\beta\gamma}{}^\rho - (-1)^{|\beta||\gamma|} m_{\alpha\gamma\beta}{}^\rho) O_\rho &= -Q(\lambda_{\alpha\beta\gamma} - (-1)^{|\beta||\gamma|} \lambda_{\alpha\gamma\beta}). \end{aligned} \quad (3.6)$$

Another consistency condition is that the RHS of (3.4) should be graded symmetric for the all 3 indices α, β, γ since $\mathbf{O}_{\alpha\beta\gamma}$ has such property.

We claim that

Claim (1). $\mathbf{K} \mathbf{C}_{\alpha\beta\gamma} = \hbar \mathbf{F}_{\alpha\beta\gamma}$, where

$$\mathbf{F}_{\alpha\beta\gamma} := -(-1)^{|\alpha|} \{(\mathbf{O}_\alpha, \mathbf{O}_{\beta\gamma})_{\hbar} + (-1)^{|\beta|} (\mathbf{O}_{\alpha\beta}, \mathbf{O}_\gamma)_{\hbar} - (-1)^{(|\alpha|+1)(|\beta|+1)} (\mathbf{O}_\beta, \mathbf{O}_{\alpha\gamma})_{\hbar}\},$$

so that $QC_{\alpha\beta\gamma} = 0$.

Claim (2). $\mathbf{K} \mathbf{N}_{\alpha\beta\gamma} + (m_{\beta\gamma}{}^\rho m_{\alpha\rho}{}^\sigma - (-1)^{|\alpha||\beta|} m_{\alpha\gamma}{}^\rho m_{\beta\rho}{}^\sigma) \mathbf{O}_\sigma = -\hbar \mathbf{C}_{[\alpha\beta]\gamma}$, where

$$\begin{aligned} \mathbf{C}_{[\alpha\beta]\gamma} &:= \mathbf{C}_{\alpha\beta\gamma} - (-1)^{|\alpha||\beta|} \mathbf{C}_{\beta\alpha\gamma}, \\ \mathbf{N}_{\alpha\beta\gamma} &:= m_{\beta\gamma}{}^\rho \boldsymbol{\lambda}_{\alpha\rho} - (-1)^{\alpha\beta} m_{\alpha\gamma}{}^\rho \boldsymbol{\lambda}_{\beta\rho} + (-1)^\alpha \mathbf{O}_\alpha \cdot \boldsymbol{\lambda}_{\beta\gamma} - (-1)^{\alpha\beta+\beta} \mathbf{O}_\beta \cdot \boldsymbol{\lambda}_{\alpha\gamma}. \end{aligned}$$

From the classical limit $QN_{\alpha\beta\gamma} + (m_{\beta\gamma}{}^\rho m_{\alpha\rho}{}^\sigma - (-1)^{|\alpha||\beta|} m_{\alpha\gamma}{}^\rho m_{\beta\rho}{}^\sigma) O_\sigma = 0$ of claim (2) we conclude that

$$m_{\beta\gamma}{}^\rho m_{\alpha\rho}{}^\sigma - (-1)^{|\alpha||\beta|} m_{\alpha\gamma}{}^\rho m_{\beta\rho}{}^\sigma = 0.$$

Hence claim (2) reduce to

$$\mathbf{K} \mathbf{N}_{\alpha\beta\gamma} = -\hbar \mathbf{C}_{[\alpha\beta]\gamma}.$$

Actually we have a stronger result:

Claim (3). There exist some $\xi_{\alpha\beta\gamma} \in \mathcal{C}[[\hbar]]^{|\alpha|+|\beta|+|\gamma|-1}$ satisfying $\mathbf{C}_{[\alpha\beta]\gamma} = \mathbf{K}\xi_{\alpha\beta\gamma}$ and

$$\begin{aligned}\xi_{0\beta\gamma} &= \xi_{\alpha 0\gamma} = \xi_{\alpha\beta 0} = 0, \\ \xi_{\alpha\beta\gamma} + (-1)^{|\alpha||\beta|} \xi_{\beta\alpha\gamma} &= 0, \\ \xi_{\alpha\beta\gamma} - (-1)^{|\beta||\gamma|} \xi_{\alpha\gamma\beta} + (-1)^{|\alpha|(|\beta|+|\gamma|)} \xi_{\beta\gamma\alpha} &= 0,\end{aligned}$$

such that the classical limit $\xi_{\alpha\beta\gamma}$ of $\xi_{\alpha\beta\gamma}$ is

$$\xi_{\alpha\beta\gamma} = \lambda_{\alpha\beta\gamma} - (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha\gamma}.$$

From the above claim we have $(m_{\alpha\beta\gamma}{}^\rho - (-1)^{|\alpha||\beta|} m_{\beta\alpha\gamma}{}^\rho) O_\rho = 0$. Hence, together with (3.6) we deduce that

$$m_{\alpha\beta\gamma}{}^\rho = (-1)^{|\alpha||\beta|} m_{\beta\alpha\gamma}{}^\rho = (-1)^{|\beta||\gamma|} m_{\alpha\gamma\beta}, \quad m_{0\beta\gamma}{}^\rho = 0$$

and

$$\lambda_{\alpha\beta\gamma} = (-1)^{|\beta||\gamma|} \lambda_{\alpha\gamma\beta}, \quad \lambda_{0\beta\gamma} = \lambda_{\alpha 0\gamma} = \lambda_{\alpha\beta 0} = 0.$$

From claim (3), the expression $\xi_{\alpha\beta\gamma} - (\lambda_{\alpha\beta\gamma} - (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha\gamma})$ is divisible by \hbar so that we can define $\eta_{\alpha\beta\gamma} \in \mathcal{C}[[\hbar]]^{|\alpha|+|\beta|+|\gamma|-1}$ by the formula

$$\hbar \eta_{\alpha\beta\gamma} := \xi_{\alpha\beta\gamma} - (\lambda_{\alpha\beta\gamma} - (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha\gamma}).$$

Define

$$\lambda_{\alpha\beta\gamma} := \lambda_{\alpha\beta\gamma} - \frac{\hbar}{3} (\eta_{\alpha\beta\gamma} + (-1)^{|\beta||\gamma|} \eta_{\alpha\gamma\beta}),$$

such that $\lambda_{\alpha\beta\gamma}|_{\hbar=0} = \lambda_{\alpha\beta\gamma}$. Then, from claim (3), we obtain that

$$\begin{aligned}\lambda_{0\beta\gamma} &= \lambda_{\alpha 0\gamma} = \lambda_{\alpha\beta 0} = 0, \\ \lambda_{\alpha\beta\gamma} - (-1)^{|\beta||\gamma|} \lambda_{\alpha\gamma\beta} &= 0, \\ \lambda_{\alpha\beta\gamma} - (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha\gamma} &= \xi_{\alpha\beta\gamma}.\end{aligned}\tag{3.7}$$

We refer the above choice for $\lambda_{\alpha\beta\gamma}$ together with the previous choice (3.3) for $\lambda_{\alpha\beta}$ "quantum gauge" (see the forthcoming remark).

Now consider the expression $\mathbf{L}_{\alpha\beta\gamma} := \mathbf{C}_{\alpha\beta\gamma} - m_{\alpha\beta\gamma}{}^\rho \mathbf{O} - \mathbf{K}\lambda_{\alpha\beta\gamma}$. Then (i) $\mathbf{L}_{\alpha\beta\gamma}$ is divisible by \hbar , (ii) $\mathbf{L}_{\alpha\beta\gamma}$ is graded symmetric for the all 3 indices α, β, γ , (iii) $\mathbf{L}_{\alpha\beta 0} = 0$. Hence we can define $\mathbf{O}_{\alpha\beta\gamma} := \frac{1}{\hbar} \mathbf{L}_{\alpha\beta\gamma}$, i.e.,

$$\hbar \mathbf{O}_{\alpha\beta\gamma} = \mathbf{C}_{\alpha\beta\gamma} - m_{\alpha\beta\gamma}{}^\rho \mathbf{O} - \mathbf{K}\lambda_{\alpha\beta\gamma},$$

such that $\mathbf{O}_{\alpha\beta\gamma} = (-1)^{|\alpha||\beta|} \mathbf{O}_{\beta\alpha\gamma} = (-1)^{|\beta||\gamma|} \mathbf{O}_{\alpha\gamma\beta}$ and $\mathbf{O}_{\alpha\beta 0} = 0$. From claim (1), it also follows that

$$\mathbf{K}\mathbf{O}_{\alpha\beta\gamma} = -(-1)^{|\alpha|} \left\{ (\mathbf{O}_\alpha, \mathbf{O}_{\beta\gamma})_\hbar + (-1)^{|\beta|} (\mathbf{O}_{\alpha\beta}, \mathbf{O}_\gamma)_\hbar - (-1)^{(|\alpha|+1)(|\beta|+1)} (\mathbf{O}_\beta, \mathbf{O}_{\alpha\gamma})_\hbar \right\}.$$

Let

$$\begin{aligned}\Theta &:= t^\alpha + \frac{1}{2}t^\beta t^\alpha \mathbf{O}_{\alpha\beta} + \frac{1}{3!}t^\gamma t^\beta t^\alpha \mathbf{O}_{\alpha\beta\gamma} \bmod t_H^4, \\ A_{\beta\gamma}^\gamma &:= m_{\beta\gamma}^\rho + t^\alpha m_{\alpha\beta\gamma}^\rho \bmod t_H^2, \\ \Lambda_{\beta\gamma} &:= \lambda_{\alpha\beta} + t^{\tilde{\alpha}} \lambda_{\alpha\beta\gamma} \bmod t_H^2.\end{aligned}$$

Then we just have solved the quantum master equation modulo t_H^2 :

$$\hbar \partial_\beta \partial_\gamma \Theta = \partial_\beta \Theta \cdot \partial_\gamma \Theta - A_{\beta\gamma}^\rho \mathbf{O}_\rho + \mathbf{K} \Lambda_{\alpha\beta\gamma} + (\Theta, \Lambda_{\alpha\beta\gamma})_{\hbar} \bmod t_H^3,$$

such that

$$\begin{aligned}\partial_0 \Theta &= 1 \bmod t_H^3, \\ \mathbf{K} \Theta + \frac{1}{2}(\Theta, \Theta)_{\hbar} &= 0 \bmod t_H^4.\end{aligned}$$

and

$$\begin{aligned}A_{\beta\gamma}^\rho - (-1)^{|\beta||\gamma|} A_{\gamma\beta}^\rho &= 0 \bmod t_H^2, \\ \partial_\alpha A_{\beta\gamma}^\rho - (-1)^{|\alpha||\beta|} \partial_\beta A_{\alpha\gamma}^\rho &= 0 \bmod t_H, \\ A_{\beta\gamma}^\sigma A_{\alpha\sigma}^\rho - (-1)^{|\alpha||\beta|} A_{\alpha\gamma}^\sigma A_{\beta\sigma}^\rho &= 0 \bmod t_H.\end{aligned}$$

Remark 3.3. The choice (3.3) we have made for $\Lambda_{\alpha\beta} = \lambda_{\alpha\beta} \bmod t_H$ and the choices (3.3) and (3.7) for $\Lambda_{\beta\gamma} = \lambda_{\beta\gamma} + t^{\tilde{\alpha}} \lambda_{\alpha\beta\gamma} \bmod t_H$ are what we call "quantum gauge". In this paper the quantum gauge is merely a convenient choice of $\Lambda_{\alpha\beta}$ in solving quantum master equation and its full meaning is the subject of the next paper in this series. To explain the quantum gauge in more details, let's sketch a proof of claim (3) in example 3.2. Consider the expression $\mathbf{N}_{\alpha\beta\gamma}$ in claim (2). We note that

$$\begin{aligned}\mathbf{N}_{0\beta\gamma} &= \mathbf{N}_{\alpha 0\gamma} = \mathbf{N}_{\alpha\beta 0} = 0, \\ \mathbf{N}_{\alpha\beta\gamma} + (-1)^{|\alpha||\beta|} \mathbf{N}_{\beta\alpha\gamma} &= 0, \\ \mathbf{N}_{\alpha\beta\gamma} - (-1)^{|\beta||\gamma|} \mathbf{N}_{\alpha\gamma\beta} + (-1)^{|\alpha|(|\beta|+|\gamma|)} \mathbf{N}_{\beta\gamma\alpha} &= 0.\end{aligned}\tag{3.8}$$

As a corollary of claim (2) we have obtained that $\mathbf{K} \mathbf{N}_{\alpha\beta\gamma} = -\hbar \mathbf{M}_{[\alpha\beta]\gamma}$, which implies that $Q \mathbf{N}_{\alpha\beta\gamma} = 0$, where $N_{\alpha\beta\gamma}$ is the classical limit of $\mathbf{N}_{\alpha\beta\gamma}$ given by

$$N_{\alpha\beta\gamma} := m_{\beta\gamma}^\rho \lambda_{\alpha\rho} - (-1)^{\alpha\beta} m_{\alpha\gamma}^\rho \lambda_{\beta\rho} + (-1)^\alpha O_\alpha \cdot \lambda_{\beta\gamma} - (-1)^{\alpha\beta+\beta} O_\beta \cdot \lambda_{\alpha\gamma}.$$

Hence we have

$$N_{\alpha\beta\gamma} = n_{\alpha\beta\gamma}^\rho O_\gamma + Q x_{\alpha\beta\gamma},\tag{3.9}$$

for unique $n_{\alpha\beta\gamma}^\rho \in \mathbb{k}$ and some $x_{\alpha\beta\gamma} \in \mathcal{C}^{|\alpha|+|\beta|+|\gamma|-2}$ defined modulo $\text{Ker } Q$. From (3.8) we deduce that

$$\begin{aligned}n_{0\beta\gamma} &= n_{\alpha 0\gamma} = n_{\alpha\beta 0} = 0, \\ n_{\alpha\beta\gamma} + (-1)^{|\alpha||\beta|} n_{\beta\alpha\gamma} &= 0, \\ n_{\alpha\beta\gamma} - (-1)^{|\beta||\gamma|} n_{\alpha\gamma\beta} + (-1)^{|\alpha|(|\beta|+|\gamma|)} n_{\beta\gamma\alpha} &= 0,\end{aligned}$$

and we can make a choice for $x_{\alpha\beta\gamma}$ such that

$$\begin{aligned} x_{0\beta\gamma} &= x_{\alpha 0\gamma} = x_{\alpha\beta 0} = 0, \\ x_{\alpha\beta\gamma} + (-1)^{|\alpha||\beta|} x_{\beta\alpha\gamma} &= 0, \\ x_{\alpha\beta\gamma} - (-1)^{|\beta||\gamma|} x_{\alpha\gamma\beta} + (-1)^{|\alpha|(|\beta|+|\gamma|)} x_{\beta\gamma\alpha} &= 0. \end{aligned}$$

From (3.9), we deduce that the expression $\mathbf{N}_{\alpha\beta\gamma} - n_{\alpha\beta\gamma}{}^\rho \mathbf{O}_\gamma - \mathbf{K}x_{\alpha\beta\gamma}$ is divisible by \hbar , so that we define $\xi_{\alpha\beta\gamma} \in \mathcal{C}[[\hbar]]^{|\alpha|+|\beta|+|\gamma|-1}$ by the formula

$$\hbar \xi_{\alpha\beta\gamma} := \mathbf{N}_{\alpha\beta\gamma} - n_{\alpha\beta\gamma}{}^\rho \mathbf{O}_\gamma - \mathbf{K}x_{\alpha\beta\gamma},$$

and all the properties of $\xi_{\alpha\beta\gamma}$ listed in claim (3) follows. Then, from the 3rd relation in (3.7), we have $\hbar (\lambda_{\alpha\beta\gamma} - (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha\gamma}) = \mathbf{N}_{\alpha\beta\gamma} - n_{\alpha\beta\gamma}{}^\rho \mathbf{O}_\gamma - \mathbf{K}x_{\alpha\beta\gamma}$. More explicitly, we have

$$\begin{aligned} \hbar (\lambda_{\alpha\beta\gamma} - (-1)^{|\alpha||\beta|} \lambda_{\beta\alpha\gamma}) &= m_{\beta\gamma}{}^\rho \lambda_{\alpha\rho} - (-1)^{\alpha\beta} m_{\alpha\gamma}{}^\rho \lambda_{\beta\rho} \\ &\quad + (-1)^\alpha \mathbf{O}_\alpha \cdot \lambda_{\beta\gamma} - (-1)^{\alpha\beta+\beta} \mathbf{O}_\beta \cdot \lambda_{\alpha\gamma} \\ &\quad - n_{\alpha\beta\gamma}{}^\rho \mathbf{O}_\gamma - \mathbf{K}x_{\alpha\beta\gamma}. \end{aligned}$$

Now consider the definition $\Lambda_{\beta\gamma} = \lambda_{\alpha\beta} + t^{\bar{\alpha}} \lambda_{\alpha\beta\gamma} \bmod t_H^2$. Then quantum gauge conditions (3.3) and (3.7) are summarized as follows

$$\begin{aligned} \Lambda_{\beta 0} &= 0 \bmod t_H^2, \\ \Lambda_{a_2 a_1} &= (-1)^{|a_2||a_1|} \Lambda_{a_1 a_2} \bmod t_H^2, \\ \partial_{\bar{\alpha}} \Lambda_{\beta\gamma} &= 0 \bmod t_H, \\ \hbar (\partial_{\bar{\alpha}} \Lambda_{\beta\gamma} - (-1)^{|\alpha||\beta|} \partial_{\bar{\beta}} \Lambda_{\alpha\gamma}) &= -A_{\beta\gamma}{}^\rho \Lambda_{\alpha\rho} + (-1)^{|\alpha||\beta|} A_{\alpha\gamma}{}^\rho \Lambda_{\beta\rho} \\ &\quad - (-1)^{|\alpha|} \partial_{\bar{\alpha}} \Theta \cdot \Lambda_{\beta\gamma} + (-1)^{|\alpha||\beta|+|\beta|} \partial_{\bar{\beta}} \Theta \cdot \Lambda_{\alpha\gamma} \\ &\quad - B_{\alpha\beta\gamma}{}^\rho \partial_{\bar{\rho}} \Theta - \mathbf{K}X_{\alpha\beta\gamma} - (\Theta, X_{\alpha\beta\gamma})_{\hbar} \\ &\quad \bmod t_H, \end{aligned}$$

where $B_{\alpha\beta\gamma}{}^\rho = n_{\alpha\beta\gamma}{}^\rho \bmod t_H$ and $X_{\alpha\beta\gamma} = x_{\alpha\beta\gamma} \bmod t_H$. In general we are going to solve the quantum master equation with quantum gauge choice for $\Lambda_{\alpha\beta}$, which is the above condition without modulo t_H^2 or t_H .

Remark 3.4. We note that there is some ambiguity in solutions for Θ and $\Lambda_{\alpha\beta}$. There are two sources of ambiguity. With the fixed initial condition $\Theta = t^{\bar{\alpha}} \mathbf{f}(e_{\alpha}) \bmod t_H$, namely the quantization map $\mathbf{f}, \lambda_{\alpha\beta} = \Lambda_{\alpha\beta} \bmod t_H$ is not uniquely determined, which effects $\mathbf{O}_{\alpha\beta}$ in $\Theta = t^{\bar{\alpha}} \mathbf{O}_{\alpha} + \frac{1}{2} t^{\bar{\beta}} t^{\bar{\alpha}} \mathbf{O}_{\alpha\beta} \bmod t_H^2$ etc. We may also vary the initial condition within the same quantum homotopy class $\mathbf{f} \sim \mathbf{f}' = \mathbf{f} + \mathbf{K}\mathbf{s}$ such that $\mathbf{f}'(e_0) = \mathbf{f}(e_0) =$

1. Hence the above theorem does not necessarily imply that the 3-tensor $A_{\alpha\beta}^\gamma$ is independent of those ambiguities. We claim, however, that every possible ambiguity does not effect the the 3-tensor $A_{\alpha\beta}^\gamma$: this claim implies that quantum correlation functions are quantum homotopy invariants. We shall need to introduce notion of quantum descendant homotopy, which deserves a separate consideration to appear elsewhere, to establish the above claim.

Remark 3.5. Consider the coefficients of expansion of $A_{\alpha\beta}^\gamma$ at $t_H = 0$:

$$A_{\alpha\beta}^\gamma = m_{\alpha\beta}^\sigma + \sum_{\rho} t^\rho m_{\rho\alpha\beta}^\sigma + \frac{1}{2!} \sum_{\rho_1, \rho_2} t^{\rho_2} t^{\rho_1} m_{\rho_1\rho_2\alpha\beta}^\gamma + \cdots$$

The two conditions for graded symmetry of $A_{\alpha\beta}^\gamma$, then, imply that $m_{\alpha_1 \dots \alpha_n}^\gamma$ is totally graded symmetric for all the lower indices for all $n = 2, 3, \dots$. Hence there is a sequence m_2, m_3, m_4, \dots of graded symmetric products of ghost number zero on H , $m_n : S^n H \rightarrow H$, such that $m_n(e_{\alpha_1}, \dots, e_{\alpha_n}) = m_{\alpha_1 \dots \alpha_n}^\gamma$. The identity $A_{0\beta}^\gamma = \delta_\beta^\gamma$ imply that $m_2(e_0, e_\alpha) = e_\alpha$ and $m_n(e_0, e_{\alpha_2}, \dots, e_{\alpha_{n-1}}) = 0$ for all $n = 3, 4, 5, \dots$. A similar structure on H was discussed in section 4 of the previous paper [1] with a different presentation of quantum master equation. Once the claim in remark 3.4 is established it is trivial to show that those two structures is identical. What was not clear in the previous paper is the associativity (an easy part of proof in this paper) of the $A_{\alpha\beta}^\gamma$ summarizing an infinite sequence of relations among the sequence m_2, m_3, m_4, \dots of multilinear products. The solution in the pervious paper is suffice to determine quantum correlation functions. The new solution in this paper with quantum gauge is a preparation to define and study homotopy quantum correlation functions in the forthcoming paper.

3.1. Quantum coordinates and linear pencil of torsion-free flat connection on \mathcal{M} .

One of the immediate consequence of our main result is that that the classical limit Θ of Θ , i.e., $\Theta = \Theta|_{\hbar=0}$:

$$\Theta = t^\alpha O_\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} t^{\alpha_n} \cdots t^{\alpha_1} O_{\alpha_1 \dots \alpha_n} \in \mathcal{C}[[t_H]]^0,$$

where $O_{\alpha_1 \dots \alpha_n} = \mathbf{O}_{\alpha_1 \dots \alpha_n}|_{t=0}$, is a distinguished versal solution to the Maurer-Cartan equation of the classical limit $(\mathcal{C}, Q, (,))$ of the quantum descendant DG0LA:

$$Q\Theta + \frac{1}{2}(\Theta, \Theta) = 0. \quad (3.10)$$

- It is a versal solution since the cohomology classes of $\{O_\alpha\}$ form a basis of H - the cohomology of the complex (\mathcal{C}, Q) . It follows that the natural extended moduli space \mathcal{M} of solutions modulo the natural equivalence is smooth - minimal L_∞ -structure on H is trivial and is quasi-isomorphic to the DG0LA $(\mathcal{C}, Q, (\bullet, \bullet))$ as an L_∞ -algebra. (See [14] for a comprehensive and lucid introduction of L_∞ -algebra and morphism and references therein.)
- It is a distinguished solution since not every versal solution of (3.10) arises as the classical limit of solution of quantum master equation, i.e., the special solution to the quantum descendant equation. Equivalently it is a distinguished quasi-isomorphism of L_∞ -algebras.

Any versal solution to (3.10) identify \mathcal{M} with the affine space H with the affine coordinates $t_H = \{t^\alpha\}$ and give a coordinates system on \mathcal{M} . We call the coordinates system on \mathcal{M} induced by the distinguished solution Θ quantum coordinates on \mathcal{M} (around the base point of \mathcal{M}). To the tangent space to \mathcal{M} , our distinguished solution induces a linear pencil of connection $\nabla_h = -\hbar d + A$, where $d := dt^\alpha \frac{\partial}{\partial t^\alpha}$ and $\{A\}_\beta^\gamma := dt^\alpha A_{\alpha\beta}^\gamma$, which is torsion-free, $A_{\beta\gamma}^\sigma = (-1)^{|\beta||\gamma|} A_{\gamma\beta}^\sigma$, and flat $\nabla_h^2 = 0$:

$$dA_\beta^\gamma = A_\beta^\rho A_\rho^\gamma = 0,$$

by combining the *graded symmetry* and the *associativity relation*.

Remark 3.6. The MC equation (3.10) implies that $Q_\Theta := Q + (\Theta, \cdot)$ satisfies $Q_\Theta^2 = 0$, the Jacobi-identity for the bracket (\cdot, \cdot) implies that Q_Θ is a derivation of the bracket. Thus we have a distinguished family $(\mathcal{C}[[t_H]], Q_\Theta, (\cdot, \cdot))$ of DGLAs. Furthermore the super-Poisson law implies that Q_Θ is a derivation of the product. Thus the quadruple

$$(\mathcal{C}[[t_H]], Q_\Theta, \cdot, (\cdot, \cdot))$$

is a differential 0-algebra. Applying $\frac{\partial}{\partial t^\alpha}$ to the MC equation (3.10), we obtain that

$$Q\Theta_\alpha + (\Theta, \Theta_\alpha) = 0 \iff Q_\Theta \Theta_\alpha = 0.$$

where $\Theta_\alpha := \partial_\alpha \Theta$. In Appendix B, we shall establish that $\{\Theta_\alpha\}$ is a set of representative of a basis of the cohomology \mathfrak{H} of the cochain complex $(\mathcal{C}[[t_H]], Q_\Theta)$ such that

$$\Theta_\alpha \cdot \Theta_\beta = A_{\alpha\beta}^\gamma \Theta_\gamma + Q_\Theta \Lambda_{\alpha\beta}, \quad (3.11)$$

for unique 3-tensor $A_{\alpha\beta}^\gamma$ in $\mathbb{k}[[t_H]]$ and for some $\Lambda_{\alpha\beta} \in (\mathcal{C}[[t_H]])^{|\alpha|+|\beta|-1}$, which is defined modulo $\text{Ker } Q_\Theta$. Then, the graded commutativity and the associativity of the product \cdot imply that \mathfrak{H} is a super-commutative and associative $\mathbb{k}[[t_H]]$ -algebra with the structure "constants" $A_{\alpha\beta}^\gamma$, that is, $A_{\alpha\beta}^\gamma = (-1)^{|\alpha||\beta|} A_{\beta\alpha}^\gamma$ and $A_{\alpha\beta}^\rho A_{\rho\gamma}^\sigma = A_{\beta\gamma}^\rho A_{\alpha\rho}^\sigma$.

Note that (3.11) is the classical limit of the quantum master equation. Forgotten its quantum origin the potentiality $\partial_\alpha A_{\beta\gamma}^\sigma - (-1)^{|\alpha||\beta|} \partial_\beta A_{\alpha\gamma}^\sigma = 0$ is obscured.

Remark 3.7. Note that we can repeat the similar story as the above remark for an arbitrary versal solution $\Theta' = t^\alpha O'_\alpha + \frac{1}{2} t^\beta t^\alpha O'_{\alpha\beta} + \dots$ to (3.10) such that $\{O'_\alpha\}$ is an another set of representative of the same basis $\{e_\alpha\}$ of the cohomology H . Then the result proved in Appendix B does not necessarily imply that we have the same 3-tensor $A_{\alpha\beta}^\gamma$. It only implies that there is unique 3-tensor $A'_{\alpha\beta}{}^\gamma$ with respect to the solution Θ' such that

$$\Theta'_\alpha \cdot \Theta'_\beta = A'_{\alpha\beta}{}^\gamma \Theta'_\gamma + Q_\Theta A'_{\alpha\beta}, \quad (3.12)$$

for some $A'_{\alpha\beta} \in (\mathcal{C}[[t_H]])^{|\alpha|+|\beta|-1}$, which is defined modulo $\text{Ker } Q_{\Theta'}$. The only relation between $A_{\alpha\beta}^\gamma$ and $A'_{\alpha\beta}{}^\gamma$ is that $A'_{\alpha\beta}{}^\gamma = A_{\alpha\beta}^\gamma \bmod t_H$. It can be, actually, shown that the difference between $A_{\alpha\beta}^\gamma$ and $A'_{\alpha\beta}{}^\gamma$ in higher order is arbitrary (see the next remark). Note also that the associativity and the commutativity of the product \cdot do imply that $A'_{\alpha\beta}{}^\gamma = (-1)^{|\alpha||\beta|} A'_{\beta\alpha}{}^\gamma$ and $A'_{\alpha\beta}{}^\rho A'_{\rho\gamma}{}^\sigma = A'_{\beta\gamma}{}^\rho A'_{\alpha\rho}{}^\sigma$, while there is absolutely no reason to expect that $\partial_\alpha A'_{\beta\gamma}{}^\sigma - (-1)^{|\alpha||\beta|} \partial_\beta A'_{\alpha\gamma}{}^\sigma = 0$.

Remark 3.8. Here is a brief comparison: Modulo t_H , (3.12) is

$$O'_\alpha \cdot O'_\beta = m'_{\alpha\beta}{}^\gamma O'_\gamma + Q \lambda'_{\alpha\beta} \quad (3.13)$$

where $\lambda'_{\alpha\beta}$ is defined modulo $\xi'_{\alpha\beta}$ satisfying $Q \xi'_{\alpha\beta} = 0$. Modulo t_H^2 , (3.12) is the above equation together with the following

$$O'_{\alpha\beta} \cdot O'_\gamma + (-1)^{|\alpha||\beta|} O'_\alpha \cdot O'_{\beta\gamma} - m'_{\beta\gamma}{}^\rho O'_{\alpha\rho} - (O'_\alpha, \lambda'_{\beta\gamma}) = m'_{\alpha\beta\gamma}{}^\sigma O'_\sigma + Q \lambda'_{\alpha\beta\gamma} \quad (3.14)$$

where $\lambda'_{\alpha\beta\gamma}$ is defined modulo $\xi'_{\alpha\beta\gamma}$ satisfying $Q \xi'_{\alpha\beta\gamma} + (-1)^{|\alpha|} (O'_\alpha, \xi'_{\beta\gamma}) = 0$. Then the only content of proposition B.1 to this order is that the cohomology class of the LHS of (3.14), thus the structure constant $m'_{\alpha\beta\gamma}{}^\sigma$, does not depend on the ambiguity of $\lambda'_{\alpha\beta}$. Now we compare (3.11) with (3.12) for the leading two terms. In general O'_α differs from O_α at most by $Q \lambda_\alpha$ for some $\lambda_\alpha \in \mathcal{C}^{|\alpha|-1}$. It follows that $[O'_\alpha \cdot O'_\beta] = [O_\alpha \cdot O_\beta]$ such that $m_{\alpha\beta}{}^\gamma = m'_{\alpha\beta}{}^\gamma$. For simplicity set $O'_\alpha = O_\alpha$. Let, for example, $O'_{\alpha\beta} = O_{\alpha\beta} + b_{\alpha\beta}{}^\gamma O_\gamma$, where $\{b_{\alpha\beta}{}^\gamma\}$ is arbitrary. Then $\Theta' = t^\alpha O_\alpha + \frac{1}{2} t^\beta t^\alpha O'_{\alpha\beta} \bmod t_H^3$ solves the MC equation (3.10) modulo t_H^3 since $\Theta = t^\alpha O_\alpha + \frac{1}{2} t^\beta t^\alpha O_{\alpha\beta} \bmod t_H^3$ does. On the other hand, the LHS of (3.14) is

$$\begin{aligned} & O'_{\alpha\beta} \cdot O'_\gamma + (-1) O'_\alpha \cdot O'_{\beta\gamma} - m'_{\beta\gamma}{}^\rho O'_{\alpha\rho} - (O'_\alpha, \lambda'_{\beta\gamma}) \\ &= C_{\alpha\beta\gamma} + (b_{\alpha\beta}{}^\rho m_{\rho\gamma}{}^\sigma (-1)^{|\alpha||\beta|} b_{\beta\gamma}{}^\rho m_{\alpha\rho}{}^\sigma - m_{\beta\gamma}{}^\rho b_{\alpha\rho}{}^\sigma) O_\sigma \bmod \text{Im } Q, \end{aligned}$$

where $C_{\alpha\beta\gamma} = O_{\alpha\beta} \cdot O_\gamma + (-1) O_\alpha \cdot O_{\beta\gamma} - m_{\beta\gamma}{}^\rho O_{\alpha\rho} - (O_\alpha, \lambda_{\beta\gamma})$ such that the word-length 2 in t_H term of (3.11) is $C_{\alpha\beta\gamma} = m_{\alpha\beta\gamma}{}^\sigma O_\sigma + Q \lambda_{\alpha\beta\gamma}$. Hence the difference between $m_{\alpha\beta\gamma}{}^\sigma$ and $m'_{\alpha\beta\gamma}{}^\sigma$ is arbitrary.

3.2. Quantum coordinates and generating function of quantum correlations for the family

Consider the solution $\Theta = t^\alpha \mathbf{O}_\alpha + \frac{1}{2} t^\beta t^\alpha \mathbf{O}_{\alpha\beta} + \dots$ of quantum master equation, which implies that Θ also solve the quantum descendant equation $\mathbf{K}\Theta + \frac{1}{2}(\Theta, \Theta)_\hbar = 0$. Note that quantum descendant equation is equivalent to

$$\mathbf{K}e^{-\Theta/\hbar} = 0, \quad (3.15)$$

due to the identity $\hbar^2 e^{\Theta/\hbar} \mathbf{K}e^{-\Theta/\hbar} = \mathbf{K}\Theta + \frac{1}{2}(\Theta, \Theta)_\hbar$. Define $\Pi_{\alpha_1 \dots \alpha_n} \in \mathcal{C}[[t_H, \hbar]]^{| \alpha_1 | + \dots + | \alpha_n |}$ for $\forall n = 1, 2, \dots$ by the formula

$$\Pi_{\alpha_1 \dots \alpha_n} = (-\hbar)^n \partial_{\alpha_1} \dots \partial_{\alpha_n} e^{-\Theta/\hbar}. \quad (3.16)$$

Then quantum descendant equation (3.15) implies that

$$\mathbf{K}_\Theta \Pi_{\alpha_1 \dots \alpha_n} = 0, \quad (3.17)$$

due to the identity that $-\hbar e^{\Theta/\hbar} \mathbf{K}(\mathbf{Y} e^{-\Theta/\hbar}) = \mathbf{K}_\Theta \mathbf{Y}$ for any $\mathbf{Y} \in \mathcal{C}[[t_H, \hbar]]$:

$$\begin{aligned} \mathbf{K}(\mathbf{Y} \cdot e^{-\Theta/\hbar}) &= e^{-\Theta/\hbar} \left(\mathbf{K}\mathbf{Y} + (\Theta, \mathbf{Y})_\hbar \right) + (-1)^{|\mathbf{Y}|} \mathbf{Y} \cdot \mathbf{K}e^{-\Theta/\hbar} \\ &= e^{-\Theta/\hbar} \left(\mathbf{K}\mathbf{Y} + (\Theta, \mathbf{Y})_\hbar \right), \end{aligned}$$

where we used the definition of (2.1) for the first equality and the quantum descendant equation for the second equality. We also have $\Pi_0 = 1$ and $\Pi_{0\alpha_1 \dots \alpha_n} = \Pi_{\alpha_1 \dots \alpha_n}$ for all $\forall n \geq 1$, due to the quantum identity $\partial_0 \Theta = 1$.

We call $\Pi_{\alpha_1 \dots \alpha_n}$ quantum n -point correlators of the family (of BV QFT parametrized by \mathcal{M} in quantum coordinates). Note that

$$\begin{aligned} \Pi_\alpha &= \partial_\alpha \Theta, \\ \Pi_{\alpha\beta} &= \partial_\alpha \Theta \cdot \partial_\beta \Theta - \hbar \partial_\alpha \partial_\beta \Theta, \\ \Pi_{\alpha\beta\gamma} &= \partial_\alpha \Theta \cdot \partial_\beta \Theta \cdot \partial_\gamma \Theta - \hbar \partial_\alpha \partial_\beta \Theta \cdot \partial_\gamma \Theta - \hbar \partial_\alpha \Theta \cdot \partial_\beta \partial_\gamma \Theta - \hbar (-1)^{|\alpha||\beta|} \partial_\beta \Theta \cdot \partial_\alpha \partial_\gamma \Theta \\ &\quad + \hbar^2 \partial_\alpha \partial_\beta \partial_\gamma \Theta, \end{aligned}$$

etc.

We recall, from [1], that a solution Θ of the quantum master equation was used to define generation function of quantum correlators of the initial theory by the formula

$$e^{-\Theta/\hbar} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(-1)^n}{\hbar^n} \Theta^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\hbar^n} \Omega_n \in \mathcal{C}[[t_H]]((\hbar))^0, \quad (3.18)$$

where the sequence $\Omega_1, \Omega_2, \dots$ is defined by matching the word-lengths in t_H . Then Ω_n generates n -point quantum correlators of the initial theory:

$$\Omega_n = \frac{1}{n!} t^{\alpha_n} \dots t^{\alpha_1} \pi_{\alpha_1 \dots \alpha_n} \quad \text{where} \quad \pi_{\alpha_1 \dots \alpha_n} \in \mathcal{C}[[\hbar]]^{|\alpha_1| + \dots + |\alpha_n|},$$

such that $\mathbf{K} \pi_{\alpha_1 \dots \alpha_n} = 0$ and $\pi_{\alpha_1 \dots \alpha_n}|_{\hbar=0} = O_{\alpha_1} \dots O_{\alpha_n}$. We observe, from the definition in (3.16), that

$$\pi_{\alpha_1 \dots \alpha_n} = \Pi_{\alpha_1 \dots \alpha_n}|_{t_H=0}. \quad (3.19)$$

Fix a QFT cycle \mathbf{c} of dimension N and denote $\mathbf{c}(\mathbf{a}) = \langle \mathbf{a} \rangle$. The generating functional $\mathcal{Z}(t_H)$ of quantum correlation functions of the initial theory was defined by the formula

$$\begin{aligned} \mathcal{Z} &= \langle e^{-\Theta/\hbar} \rangle := \langle 1 \rangle + \sum_{n=1}^{\infty} \frac{(-1)^n}{\hbar^n} \langle \Omega_n \rangle \\ &= \langle 1 \rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(-1)^n}{\hbar^n} t^{\alpha_n} \dots t^{\alpha_1} \langle \pi_{\alpha_1 \dots \alpha_n} \rangle \end{aligned} \quad (3.20)$$

such that an arbitrary n -point correlation function $\langle \pi_{\alpha_1 \dots \alpha_n} \rangle$ is obtained as follows:

$$\langle \pi_{\alpha_1 \dots \alpha_n} \rangle \equiv (-\hbar)^n \partial_{\alpha_1} \dots \partial_{\alpha_n} \mathcal{Z}(t_H)|_{t=0} \in \mathbb{K}[[\hbar]].$$

Remark that \mathcal{Z} is a formal power series in t_H and formal Laurent series in \hbar .

Now we introduce the notion of quantum correlation function for the family. We use the notation $\langle \mathbf{Y} \rangle_{t_H}$ for any $\mathbf{Y} \in \mathcal{C}[[t_H, \hbar]]$ such that

$$\langle \mathbf{Y} \rangle_{t_H} = \langle \mathbf{Y} \cdot e^{-\Theta/\hbar} \rangle := \langle \mathbf{Y} \rangle + \sum_{n=1}^{\infty} \frac{(-1)^n}{\hbar^n} \langle \mathbf{Y} \cdot \Omega_n \rangle.$$

Remark that $\langle \mathbf{Y} \rangle_{t_H}$ is a formal Laurent series \hbar , while $\langle \mathbf{Y} \rangle_{t_H=0} = \langle \mathbf{Y} \rangle$ is a formal power series \hbar . Then $\langle \mathbf{K} \Theta \Xi \rangle_{t_H} = 0$ for all $\Xi \in \mathcal{C}[[t_H, \hbar]]$ since

$$\langle \mathbf{K} \Theta \Xi \cdot e^{-\Theta/\hbar} \rangle \equiv \langle \mathbf{K} (\Xi \cdot e^{-\Theta/\hbar}) \rangle := \langle \mathbf{K} \Xi \rangle + \sum_{n=1}^{\infty} \frac{(-1)^n}{\hbar^n} \langle \mathbf{K} (\Xi \cdot \Omega_n) \rangle = 0.$$

We define n -point quantum correlation functions in the family by $\langle \Pi_{\alpha_1 \dots \alpha_n} \rangle_{t_H}$ for all $n = 1, 2, 3, \dots$. Then we deduce that

$$\langle \Pi_{\alpha_1 \dots \alpha_n} \rangle_{t_H} \equiv (-\hbar)^n \partial_{\alpha_1} \dots \partial_{\alpha_n} \mathcal{Z} \in \mathbb{K}[[t_H]]((\hbar)).$$

and

$$\langle \Pi_{\alpha_1 \dots \alpha_n} \rangle_{t_H=0} = \langle \pi_{\alpha_1 \dots \alpha_n} \rangle. \quad (3.21)$$

Note that the quantum master equation

$$\hbar \partial_\beta \partial_\gamma \Theta = \partial_\beta \Theta \cdot \partial_\gamma \Theta - A_{\beta\gamma}{}^\sigma \partial_\sigma \Theta - \mathbf{K} \Lambda_{\beta\gamma} - (\Theta, \Lambda_{\beta\gamma})_{\hbar},$$

is equivalent to $\Pi_{\alpha\beta} = A_{\alpha\beta}{}^\gamma \Pi_\gamma + \mathbf{K}_\Theta \Lambda_{\alpha\beta}$, so that

$$\langle \Pi_{\alpha\beta} \rangle_{t_H} = A_{\alpha\beta}{}^\gamma \langle \Pi_\gamma \rangle_{t_H}, \quad (3.22)$$

which relates quantum 2-point functions to quantum 1-point correlation functions both in the family. For higher correlations, it is useful to consider the following form of the quantum master equation:

$$\left(\hbar^2 \partial_\beta \partial_\gamma + \hbar A_{\beta\gamma}{}^\rho \partial_\rho \right) e^{-\Theta/\hbar} = \mathbf{K} \left(\Lambda_{\alpha\beta} e^{-\Theta/\hbar} \right), \quad (3.23)$$

since

$$\begin{aligned} LHS: \left(\hbar^2 \partial_\beta \partial_\gamma + \hbar A_{\beta\gamma}{}^\rho \partial_\rho \right) e^{-\Theta/\hbar} &= e^{-\Theta/\hbar} \left(\partial_\beta \Theta \cdot \partial_\gamma \Theta - \hbar \partial_\beta \partial_\gamma \Theta - A_{\beta\gamma}{}^\rho \partial_\rho \Theta \right), \\ RHS: \mathbf{K} \left(\Lambda_{\alpha\beta} e^{-\Theta/\hbar} \right) &= e^{-\Theta/\hbar} \left(\mathbf{K} \Lambda_{\alpha\beta} + (\Theta, \Lambda_{\beta\gamma})_{\hbar} \right). \end{aligned}$$

Similarly, the quantum identity $\partial_0 \Theta = 1$ is equivalent to

$$-\hbar \partial_0 e^{-\Theta/\hbar} = e^{-\Theta/\hbar}. \quad (3.24)$$

Note the relations (3.23) and (3.24) after using $\langle \mathbf{K} \left(\Lambda_{\alpha\beta} e^{-\Theta/\hbar} \right) \rangle = 0$ imply that

Lemma 3.1. *The generating function $\mathcal{Z} \in \mathbb{k}[[t_H]]((\hbar))$ satisfies the following system of differential equations:*

$$\begin{aligned} \left(\hbar \frac{\partial^2}{\partial t^\alpha \partial t^\beta} + A_{\alpha\beta}{}^\gamma \frac{\partial}{\partial t^\gamma} \right) \mathcal{Z} &= 0, \\ \left(\hbar \frac{\partial}{\partial t^0} + 1 \right) \mathcal{Z} &= 0, \\ \left(|e_\alpha| t^\alpha \frac{\partial}{\partial t^\alpha} - N \right) \mathcal{Z} &= 0, \end{aligned}$$

where N is the dimension of QFT cycle.

Proof. The first two relations are consequences of (3.23) and (3.24), respectively. The last relation is a direct consequence of $\langle \mathbf{Y} \rangle = 0$ for all $\mathbf{Y} \in \mathcal{C}[[\hbar]]$ with ghost number not equals to N . Consider the definition \mathcal{Z} in (3.20):

$$\mathcal{Z} = \langle e^{-\Theta/\hbar} \rangle = \langle 1 \rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(-1)^n}{\hbar^n} \sum_{\alpha_1, \dots, \alpha_n} t^{\alpha_n} \dots t^{\alpha_1} \langle \pi_{\alpha_1 \dots \alpha_n} \rangle,$$

where we temporarily abandon Einstein summation convention. Then the relation follows from the identity

$$\sum_{\alpha} |e_{\alpha}| t^{\alpha} \frac{\partial}{\partial t^{\alpha}} (t^{\alpha_n} \dots t^{\alpha_1} \langle \pi_{\alpha_1 \dots \alpha_n} \rangle) = \left(\sum_{i=1}^n |e_{\alpha_i}| \right) t^{\alpha_n} \dots t^{\alpha_1} \langle \pi_{\alpha_1 \dots \alpha_n} \rangle$$

and the condition that

$$t^{\alpha_n} \dots t^{\alpha_1} \langle \pi_{\alpha_1 \dots \alpha_n} \rangle = 0$$

for $|\pi_{\alpha_1 \dots \alpha_n}| \equiv -\sum_{i=1}^n |t^{\alpha_i}| \equiv \sum_{i=1}^n |e_{\alpha_i}| \neq N$.

□

Let $\{\mathbf{P}_{\alpha_1 \alpha_2}^{\gamma}, \mathbf{P}_{\alpha_1 \alpha_2 \alpha_3}^{\gamma}, \dots\}$ be the set of infinite sequences in $\mathbb{K}[[t_H, \hbar]]$ defined recursively with the initial condition $\mathbf{P}_{\alpha_1 \alpha_2}^{\gamma} = A_{\alpha_1 \alpha_2}^{\gamma}$ and, for $n = 3, 4, \dots$,

$$\mathbf{P}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\gamma} = -\hbar \partial_{\alpha_1} \mathbf{P}_{\alpha_2 \dots \alpha_n}^{\gamma} + \mathbf{P}_{\alpha_2 \dots \alpha_n}^{\rho} A_{\alpha_1 \rho}^{\gamma}.$$

We note that $\mathbf{P}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\gamma}$ is at most of degree $(n-2)$ polynomial in \hbar with coefficient in $\mathbb{K}[[t_H]]$. Note also that $\mathbf{P}_{0\beta}^{\gamma} = \delta_{\beta}^{\gamma}$ and $\mathbf{P}_{0\alpha_1 \alpha_2 \dots \alpha_n}^{\gamma} = \mathbf{P}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\gamma}$ for $\forall n \geq 2$. Then

Lemma 3.2. *Any n -point quantum correlation function $\langle \Pi_{\alpha_1 \dots \alpha_n} \rangle_{t_H}$, for $n = 2, 3, \dots$, for the family is given by the following formula*

$$\langle \Pi_{\alpha_1 \dots \alpha_n} \rangle_{t_H} = \mathbf{P}_{\alpha_1 \dots \alpha_n}^{\gamma} \langle \Pi_{\gamma} \rangle_{t_H}.$$

Proof. From the quantum master equation in the form of (3.23):

$$\hbar^2 \partial_{\beta} \partial_{\gamma} e^{-\Theta/\hbar} = -\hbar A_{\beta\gamma}^{\rho} \partial_{\rho} e^{-\Theta/\hbar} + \mathbf{K}(\Lambda_{\beta\gamma} \cdot e^{-\Theta/\hbar}).. \quad (3.25)$$

as a direct consequence of the quantum master equation. Applying $-\hbar \partial_{\alpha}$ to the both hand sides of (3.25), and using (3.25) one more time, we obtain that

$$\begin{aligned} -\hbar^3 \partial_{\alpha} \partial_{\beta} \partial_{\gamma} e^{-\Theta/\hbar} &= \left(-\hbar \partial_{\alpha} A_{\beta\gamma}^{\sigma} + A_{\beta\gamma}^{\rho} A_{\alpha\rho}^{\sigma} \right) \Theta_{\sigma} \cdot e^{-\Theta/\hbar} \\ &\quad + \mathbf{K}(\mathbf{X}_{\alpha\beta\gamma} \cdot e^{-\Theta/\hbar}), \end{aligned} \quad (3.26)$$

where

$$\mathbf{X}_{\alpha\beta\gamma} = -\hbar(-1)^{|\alpha|} \partial_{\alpha} \Lambda_{\beta\gamma} + (-1)^{|\alpha|} \Theta_{\alpha} \cdot \Lambda_{\beta\gamma} + A_{\beta\gamma}^{\rho} \mathbf{X}_{\alpha\rho}.$$

In general we deduce that, for $n = 3, 4, \dots$,

$$(-\hbar)^n \partial_{\alpha_1} \partial_{\alpha_2} \dots \partial_{\alpha_n} e^{-\Theta/\hbar} = \mathbf{P}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\sigma} \Theta_{\sigma} \cdot e^{-\Theta/\hbar} + \mathbf{K}(\mathbf{X}_{\alpha_1 \alpha_2 \dots \alpha_n} \cdot e^{-\Theta/\hbar}), \quad (3.27)$$

where

$$\begin{aligned}\mathbf{P}_{\alpha_1 \alpha_2 \dots \alpha_n}^\sigma &= -\hbar \partial_\alpha \mathbf{P}_{\alpha_2 \dots \alpha_n}^\sigma + \mathbf{P}_{\alpha_2 \dots \alpha_n}^\rho A_{\alpha_1 \rho}^\sigma, \\ \mathbf{X}_{\alpha_1 \alpha_2 \dots \alpha_n} &= \hbar (-1)^{|\alpha_1|} \partial_{\alpha_1} \mathbf{X}_{\alpha_2 \dots \alpha_n} + (-1)^{|\alpha_1|} \boldsymbol{\Theta}_{\alpha_1} \cdot \mathbf{X}_{\alpha_2 \dots \alpha_n} + \mathbf{P}_{\alpha_2 \dots \alpha_n}^\rho \mathbf{X}_{\alpha_1 \rho}\end{aligned}$$

with the initial conditions $\mathbf{P}_{\alpha_1 \alpha_2}^\gamma = A_{\alpha_1 \alpha_2}^\gamma$ and $\mathbb{X}_{\alpha_1 \alpha_2} = \mathbf{A}_{\alpha_1 \alpha_2}$. From (3.16), the relation (3.27) is equivalent to

$$\mathbf{\Pi}_{\alpha_1 \dots \alpha_n}^\rho = \mathbf{P}_{\alpha_1 \dots \alpha_n}^\gamma \mathbf{\Pi}_\gamma + \mathbf{K}_\Theta \mathbf{X}_{\alpha_1 \dots \alpha_n}, \quad (3.28)$$

which implies our lemma. \square

Now set $t_H = 0$ in the identity (3.28) and use the conditions that $\boldsymbol{\Theta}|_{t_H=0} = 0$ and $\boldsymbol{\Theta}_\alpha|_{t_H=0} = \mathbf{O}_\alpha$ as well as the relation (3.21) to obtain that

$$\boldsymbol{\pi}_{\alpha_1 \dots \alpha_n} = \mathbf{p}_{\alpha_1 \dots \alpha_n}^\gamma \mathbf{O}_\gamma + \mathbf{K} \mathbf{x}_{\alpha_1 \dots \alpha_n} \quad (3.29)$$

where $\mathbf{p}_{\alpha_1 \dots \alpha_n}^\gamma := \mathbf{P}_{\alpha_1 \dots \alpha_n}^\gamma|_{t_H=0}$ and $\mathbf{x}_{\alpha_1 \dots \alpha_n} := \mathbf{X}_{\alpha_1 \dots \alpha_n}|_{t_H=0}$. Note that $\mathbf{p}_{\alpha_1 \dots \alpha_n}^\gamma \in \mathbb{k}[[\hbar]]$ is at most of degree $(n-2)$ polynomial in \hbar with coefficients in \mathbb{k} . Hence we have

$$\langle \boldsymbol{\pi}_{\alpha_1 \dots \alpha_n} \rangle = \mathbf{p}_{\alpha_1 \dots \alpha_n}^\gamma \langle \mathbf{O}_\gamma \rangle.$$

which implies that the set $\{A_{\alpha\beta}^\gamma\}$ can be used to determine the generating function \mathcal{Z} completely once we fix every 1-point correlation function $\langle \mathbf{O}_\alpha \rangle$.

Example 3.3. The first few quantum correlators are

$$\begin{aligned}\boldsymbol{\pi}_\alpha &:= \mathbf{O}_\alpha, \\ \boldsymbol{\pi}_{\alpha_1 \alpha_2} &:= \mathbf{O}_{\alpha_1} \mathbf{O}_{\alpha_2} - \hbar \mathbf{O}_{\alpha_1 \alpha_2}, \\ \boldsymbol{\pi}_{\alpha_1 \alpha_2 \alpha_3} &= \mathbf{O}_{\alpha_1} \mathbf{O}_{\alpha_2} \mathbf{O}_{\alpha_3} - \hbar \mathbf{O}_{\alpha_1 \alpha_2} \mathbf{O}_{\alpha_3} - \hbar \mathbf{O}_{\alpha_1} \mathbf{O}_{\alpha_2 \alpha_3} - \hbar (-1)^{|\alpha_1| |\alpha_2|} \mathbf{O}_{\alpha_2} \mathbf{O}_{\alpha_1 \alpha_3} + \hbar^2 \mathbf{O}_{\alpha_1 \alpha_2 \alpha_3},\end{aligned}$$

and, just for fun,

$$\begin{aligned}\boldsymbol{\pi}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} &= \mathbf{O}_{\alpha_1} \mathbf{O}_{\alpha_2} \mathbf{O}_{\alpha_3} \mathbf{O}_{\alpha_4} - \hbar \mathbf{O}_{\alpha_1 \alpha_2} \mathbf{O}_{\alpha_3} \mathbf{O}_{\alpha_4} - \hbar \mathbf{O}_{\alpha_1} \mathbf{O}_{\alpha_2 \alpha_3} \mathbf{O}_{\alpha_4} - \hbar \mathbf{O}_{\alpha_1} \mathbf{O}_{\alpha_2} \mathbf{O}_{\alpha_3 \alpha_4} \\ &\quad - \hbar (-1)^{|\alpha_1| |\alpha_2|} \mathbf{O}_{\alpha_2} \mathbf{O}_{\alpha_1 \alpha_3} \mathbf{O}_{\alpha_4} - \hbar (-1)^{|\alpha_1| (|\alpha_2| + |\alpha_3|)} \mathbf{O}_{\alpha_2} \mathbf{O}_{\alpha_3} \mathbf{O}_{\alpha_1 \alpha_4} - \hbar (-1)^{|\alpha_2| |\alpha_3|} \mathbf{O}_{\alpha_1} \mathbf{O}_{\alpha_3} \mathbf{O}_{\alpha_2 \alpha_4} \\ &\quad + \hbar^2 \mathbf{O}_{\alpha_1 \alpha_2} \mathbf{O}_{\alpha_3 \alpha_4} + \hbar^2 (-1)^{|\alpha_2| |\alpha_3|} \mathbf{O}_{\alpha_1 \alpha_3} \mathbf{O}_{\alpha_2 \alpha_4} + \hbar^2 (-1)^{|\alpha_1| (|\alpha_2| + |\alpha_3|)} \mathbf{O}_{\alpha_2 \alpha_3} \mathbf{O}_{\alpha_1 \alpha_4} \\ &\quad + \hbar^2 \mathbf{O}_{\alpha_1} \mathbf{O}_{\alpha_2 \alpha_3 \alpha_4} + \hbar^2 (-1)^{|\alpha_1| |\alpha_2|} \mathbf{O}_{\alpha_2} \mathbf{O}_{\alpha_1 \alpha_3 \alpha_4} + \hbar^2 (-1)^{(|\alpha_1| + |\alpha_2|) |\alpha_3|} \mathbf{O}_{\alpha_3} \mathbf{O}_{\alpha_1 \alpha_2 \alpha_4} \\ &\quad + \hbar^2 \mathbf{O}_{\alpha_1 \alpha_2 \alpha_3} \mathbf{O}_{\alpha_4} - \hbar^3 \mathbf{O}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}\end{aligned}$$

etc. Expand $A_{\alpha\beta}^\gamma = m_{\alpha\beta}^\gamma + t^\rho m_{\rho\alpha\beta}^\gamma + \dots$ we have

$$\begin{aligned} \mathbf{p}_{\alpha_1\alpha_2}^\gamma &= m_{\alpha_1\alpha_2}^\gamma, \\ \mathbf{p}_{\alpha_1\alpha_2\alpha_3}^\gamma &= m_{\alpha_2\alpha_3}^\rho m_{\alpha_1\rho}^\gamma - \hbar m_{\alpha_1\alpha_2\alpha_3}^\gamma, \\ \mathbf{p}_{\alpha_1\alpha_2\alpha_3\alpha_4}^\gamma &= m_{\alpha_3\alpha_4}^\rho m_{\alpha_2\rho}^\sigma m_{\alpha_1\sigma}^\gamma \\ &\quad - \hbar \left(m_{\alpha_1\alpha_3\alpha_4}^\rho m_{\alpha_2\rho}^\gamma + m_{\alpha_3\alpha_4}^\rho m_{\alpha_1\alpha_2\rho}^\gamma + m_{\alpha_2\alpha_3\alpha_4}^\rho m_{\alpha_1\rho}^\gamma \right) \\ &\quad + \hbar^2 m_{\alpha_1\alpha_2\alpha_3\alpha_4}^\gamma, \end{aligned}$$

etc.

Combining (3.18) and (3.29), the expression $e^{-\Theta/\hbar}$ has the following expansion at t_H

$$e^{-\Theta/\hbar} = 1 - \frac{1}{\hbar} \mathbf{T}^\gamma \mathbf{O}_\gamma + \sum_{n=2}^{\infty} \frac{(-1)^n}{\hbar^n} \mathbf{K} \mathbf{x}^{[n]} \in \mathcal{C}[[t_H]](\hbar), \quad (3.30)$$

where

$$\mathbf{T}^\gamma := t^\gamma + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{(-1)^{n-1}}{\hbar^{n-1}} t^{\alpha_n} \dots t^{\alpha_1} \mathbf{p}_{\alpha_1 \dots \alpha_n}^\gamma, \quad (3.31)$$

and $\mathbf{x}^{[n]} = \frac{1}{n!} \bar{t}^{\alpha_n} \dots \bar{t}^{\alpha_1} \mathbf{x}_{\alpha_1 \dots \alpha_n}^\gamma$. Note that $\mathbf{T}^\gamma = t^\gamma \bmod t_H^2$ and a formal power series in t_H and \hbar^{-1} , since $\mathbf{p}_{\alpha_1 \dots \alpha_n}^\gamma$ is at most degree $n-2$ polynomial in \hbar .

We call $\{\mathbf{T}^\gamma\}$ the quantum coordinates for the family (in formal neighborhood of the base point o in \mathcal{M}). Note that this notion is independent to QFT cycle.

From (3.30), the generating function \mathcal{Z} can be expressed as, with respect to a QFT cycle,

$$\mathcal{Z} = \langle 1 \rangle - \frac{1}{\hbar} \mathbf{T}^\gamma \langle \mathbf{O}_\gamma \rangle. \quad (3.32)$$

Applying $-\hbar \partial_\beta$ to (3.32), we have $-\hbar \partial_\beta \mathcal{Z} = \partial_\beta \mathbf{T}^\gamma \langle \mathbf{O}_\gamma \rangle$. Combing with the relation that $-\hbar \partial_\beta \mathcal{Z} = \langle \mathbf{\Pi}_\beta \rangle_{t_H}$ we obtain a crucial formula:

$$\langle \mathbf{\Pi}_\beta \rangle_{t_H} = \frac{\partial \mathbf{T}^\gamma}{\partial t^\beta} \langle \mathbf{O}_\gamma \rangle, \quad (3.33)$$

which relates the set of expectation value of observables of the theory at the basepoint \mathcal{M} with set of expectation value of observables of theory at a general point in \mathcal{M} . From the definition of $\langle \mathbf{\Pi}_{\alpha_1 \dots \alpha_n} \rangle_{t_H}$ the following relation is obvious

$$\langle \mathbf{\Pi}_{\alpha_1 \dots \alpha_n} \rangle_{t_H} = (-\hbar)^{n-1} \frac{\partial^n \mathbf{T}^\gamma}{\partial t^{\alpha_1} \dots \partial t^{\alpha_n}} \langle \mathbf{O}_\gamma \rangle. \quad (3.34)$$

Thus the equation (3.33) determine every correlation function of the family of the theory. In particular, the relation $\langle \mathbf{\Pi}_{\alpha\beta} \rangle_{t_H} = \hbar \partial_\alpha \partial_\beta \mathcal{Z} \langle \mathbf{O}_\rho \rangle$ after combined with (3.22)

and (3.33) implies that $(\hbar \partial_\alpha \partial_\beta + A_{\alpha\beta}{}^\rho \partial_\rho) T^\gamma \langle \mathbf{O}_\gamma \rangle = 0$. Actually we have a better result than that:

Lemma 3.3. *Each T^γ , which is in $\mathbb{k}[[t_H, \hbar^{-1}]]$, satisfies the following system of formal differential equations*

$$(1) \quad \left(\hbar \frac{\partial^2}{\partial t^\alpha \partial t^\beta} + A_{\alpha\beta}{}^\rho \frac{\partial}{\partial t^\rho} \right) T^\gamma = 0,$$

$$(2) \quad \left(\hbar \frac{\partial}{\partial t^0} + 1 \right) T^\gamma = \hbar \delta_0^\gamma,$$

$$(3) \quad \left(|e_\alpha| t^\alpha \frac{\partial}{\partial t^\alpha} - |e_\gamma| \right) T^\gamma = 0.$$

Proof.

(1) From (3.30), the quantum master equation (3.23) implies that

$$(\hbar \partial_\alpha \partial_\beta + A_{\alpha\beta}{}^\rho \partial_\rho) T^\gamma \mathbf{O}_\gamma = \mathbf{K} \mathfrak{X}_{\alpha\beta}$$

for some $\mathfrak{X}_{\alpha\beta} \in \mathcal{C}[[t_H]]((\hbar))^{| \alpha | + | \beta | - 1}$. Consider the above identity modulo t_H^{n+1} and multiply \hbar^{n-1} :

$$\hbar^{n-1} (\hbar \partial_\alpha \partial_\beta + A_{\alpha\beta}{}^\rho \partial_\rho) T^\gamma \mathbf{O}_\gamma = \hbar^{n-1} \mathbf{K} \mathfrak{X}_{\alpha\beta} \mod t_H^{n+1}. \quad (3.35)$$

Then both $\hbar^{n-1} (\hbar \partial_\alpha \partial_\beta + A_{\alpha\beta}{}^\rho \partial_\rho) T^\gamma \mod t_H^{n+1}$ and $\hbar^{n-1} \mathfrak{X}_{\alpha\beta}$ have no negative power in \hbar . Now consider corollary 2.2, which state that "if $\mathbf{f}(\mathbf{x}) = \mathbf{K} \boldsymbol{\lambda}$ for some $\mathbf{x} \in H[[\hbar]]$ and $\boldsymbol{\lambda} \in \mathcal{C}[[\hbar]]$, then $\mathbf{x} = 0$ and $\mathbf{K} \boldsymbol{\lambda} = 0$ ". With our basis $\{e_\alpha\}$ of H , any $\mathbf{x} \in H[[\hbar]]$ can be expressed as $\mathbf{x} = \mathbf{b}^\gamma e_\gamma$ for some set $\{\mathbf{b}^\gamma\}$, where $\mathbf{b}^\gamma \in \mathbb{k}[[\hbar]]$. Hence $\mathbf{f}(\mathbf{x}) = \mathbf{b}^\gamma \mathbf{f}(e_\gamma) = \mathbf{b}^\gamma \mathbf{O}_\gamma$ by our convention. Consequently the equality $\mathbf{b}^\gamma \mathbf{O}_\gamma = \mathbf{K} \boldsymbol{\lambda}$ implies that $\mathbf{b}^\gamma e_\gamma = 0$, which is equivalent to $b^{(\ell)\gamma} e_\gamma = 0$ for all $\ell = 0, 1, 2, \dots$. It follows that $\mathbf{b}^\gamma = 0$ for $\forall \gamma$. The above quoted corollary is also valid for $\mathbf{x} \in H[[t_H, \hbar]]$ and $\boldsymbol{\lambda} \in \mathcal{C}[[t_H, \hbar]]$. Hence, from (3.35), we conclude that $\hbar^{n-1} (\hbar \partial_\alpha \partial_\beta + A_{\alpha\beta}{}^\rho \partial_\rho) T^\gamma = 0 \mod t_H^{n+1}$, which implies that $(\hbar \partial_\alpha \partial_\beta + A_{\alpha\beta}{}^\rho \partial_\rho) T^\gamma = 0 \mod t_H^{n+1}$. Then take $n \rightarrow \infty$ limit to conclude that $(\hbar \partial_\alpha \partial_\beta + A_{\alpha\beta}{}^\rho \partial_\rho) T^\gamma = 0$.

(2) The result follows from the quantum unity $\partial_0 \boldsymbol{\Theta} = 1$, hence $-\hbar \partial_0 e^{-\boldsymbol{\Theta}/\hbar} = e^{-\boldsymbol{\Theta}/\hbar}$, and (3.30) after adopting the similar argument as in (1).

(3) The result follows from (3.30) by matching the ghost number: $|T^\rho| = -|e_\gamma|$ since $e^{-\boldsymbol{\Theta}/\hbar}$ has ghost number 0 and $|\mathbf{O}_\gamma| = |e_\gamma|$.

□

Define

$$\mathcal{G}_\beta^\gamma := \frac{\partial T^\gamma}{\partial t^\beta} = \delta_\beta^\gamma - \frac{1}{\hbar} t^\alpha m_{\alpha\beta}^\gamma + \dots$$

and let \mathcal{G} denote the matrix over formal power series ring $\mathbb{k}[[t_H, \hbar^{-1}]]$ with \mathcal{G}_β^γ its $\beta\gamma$ entry.

Lemma 3.4. *The matrix \mathcal{G} is invertible and satisfies*

- (1) $d\mathcal{G}^{-1} \wedge d\mathcal{G} = 0$,
- (2) $\partial_0 \mathcal{G} = -\frac{1}{\hbar} \mathcal{G}$,
- (3) $|e_\rho| t^\rho \partial_\rho \mathcal{G}_\beta^\gamma = (|e_\gamma| - |e_\beta|) \mathcal{G}_\beta^\gamma$.

Proof. The matrix \mathcal{G} is invertible since its constant part is the identity matrix.

(1) From property (1) in lemma 3.3, we have $\hbar \partial_\alpha \mathcal{G}_\beta^\gamma + A_{\alpha\beta}^\rho \mathcal{G}_\rho^\gamma = 0$. It follows that

$$A_{\alpha\beta}^\gamma = -\hbar \partial_\alpha \mathcal{G}_\beta^\rho \mathcal{G}_\rho^{-1\gamma} = \hbar \mathcal{G}_\beta^{-1\rho} \partial_\alpha \mathcal{G}_\rho^\gamma \in \mathbb{k}[[t_H]].$$

Let A denote matrix valued 1-form which $\beta\gamma$ entry is $A_\beta^\gamma = dt^\alpha A_{\alpha\beta}^\gamma$. Then

$$A = \hbar \mathcal{G}^{-1} d\mathcal{G},$$

and the both conditions $dA = A^2 = 0$ reduce to $d\mathcal{G}^{-1} \wedge d\mathcal{G} = 0$.

(2) From property (2) in lemma 3.3, we have $\hbar \partial_0 \mathcal{G}_\beta^\gamma + \mathcal{G}_\beta^\gamma = 0$, which is equivalent to property (2).

(3) From property (3) in lemma 3.3, we have $|e_\rho| t^\rho \mathcal{G}_\rho^\gamma - |e_\gamma| T^\gamma = 0$. Applying ∂_β to the relation we obtain that $(|e_\sigma| t^\sigma \partial_\sigma - (|e_\gamma| - |e_\beta|)) \mathcal{G}_\beta^\gamma = 0$. \square

Corollary 3.1. *Let $A_{\alpha\beta}^\gamma := -\hbar \partial_\alpha \mathcal{G}_\beta^\rho \mathcal{G}_\rho^{-1\gamma}$. Then the 3-tensor $A_{\alpha\beta}^\gamma$ is in formal power series of t_H independent to \hbar and satisfies*

$$\begin{aligned} A_{\alpha\beta}^\gamma &= (-1)^{|\alpha||\beta|} A_{\beta\alpha}^\gamma, \\ \partial_\alpha A_{\beta\gamma}^\rho &= (-1)^{|\alpha||\beta|} \partial_\beta A_{\alpha\gamma}^\rho, \\ A_{\alpha\beta}^\sigma A_{\sigma\gamma}^\gamma &= A_{\beta\gamma}^\sigma A_{\alpha\sigma}^\rho, \\ A_{0\beta}^\gamma &= \delta_\beta^\gamma, \\ |e_\rho| t^\rho \partial_\rho A_{\alpha\beta}^\gamma &= (|e_\gamma| - |e_\beta| - |e_\alpha|) A_{\alpha\beta}^\gamma. \end{aligned}$$

3.2.1. Free energy as generating function of morphism of QFT algebra. This subsection is about some preliminary understanding of underlying algebraic structures of QFT as is uncovered by this paper. We consider an unital BV QFT, whose partition function is normalizable to 1. We shall see that the free energy is another avatar of the notion of QFT algebra morphism.

Consider a BV QFT with QFT cycle $\mathbf{c} : \mathcal{C} \rightarrow \mathbb{K}[[\hbar]]$ and suppose that $|\mathbf{c}| = 0$, and the partition function $\mathbf{c}(\mathbf{f}(e_0)) = \mathbf{c}(1) = \langle 1 \rangle$ is normalizable to 1. We define free energy \mathbf{F} as follows:

$$e^{-\mathbf{F}/\hbar} = 1 - \frac{1}{\hbar} \mathbf{T}^\gamma \langle \mathbf{O}_\gamma \rangle, \quad (3.36)$$

where $\langle \mathbf{O}_\alpha \rangle \in \mathbb{K}[[\hbar]]$ is normalized expectation value. Note that $\mathbf{F}|_{t_H=0} = 0$. Applying $-\hbar \partial_\gamma$ to (3.36), we have

$$-\hbar \partial_\gamma e^{-\mathbf{F}/\hbar} = \partial_\gamma \mathbf{F} \cdot e^{-\mathbf{F}/\hbar} = \partial_\gamma \mathbf{T}^\sigma \langle \mathbf{O}_\gamma \rangle = \langle \mathbf{\Pi}_\gamma \rangle_{t_H}.$$

It follows that

$$\varphi_\alpha := \partial_\alpha \mathbf{F}|_{t_H=0} = \langle \mathbf{O}_\alpha \rangle \in \mathbb{K}[[\hbar]].$$

Applying $\hbar^2 \partial_\beta \partial_\beta$ to (3.36) and using the above, we also have

$$\begin{aligned} (-\hbar)^2 \partial_\beta \partial_\beta e^{-\mathbf{F}/\hbar} &= (-\hbar \partial_\beta \partial_\gamma \mathbf{F} + \partial_\beta \mathbf{F} \cdot \partial_\gamma \mathbf{F}) e^{-\mathbf{F}/\hbar} \\ &= A_{\beta\gamma}{}^\sigma \langle \mathbf{\Pi}_\sigma \rangle_{t_H} \\ &= A_{\beta\gamma}{}^\sigma \partial_\sigma \mathbf{F} \cdot e^{-\mathbf{F}/\hbar}. \end{aligned} \quad (3.37)$$

Hence we obtain the following system of differential equations:

$$\begin{aligned} \hbar \partial_\alpha \partial_\beta \mathbf{F} &= \partial_\alpha \mathbf{F} \cdot \partial_\beta \mathbf{F} - A_{\alpha\beta}{}^\gamma \partial_\gamma \mathbf{F}, \\ \partial_0 \mathbf{F} &= 1, \end{aligned} \quad (3.38)$$

which should be compared with the quantum master equation with unity:

$$\begin{aligned} \hbar \partial_\alpha \partial_\beta \mathbf{\Theta} &= \partial_\alpha \mathbf{\Theta} \cdot \partial_\beta \mathbf{\Theta} - A_{\alpha\beta}{}^\gamma \partial_\gamma \mathbf{\Theta} - \mathbf{K} \Lambda_{\alpha\beta} - (\mathbf{\Theta}, \Lambda_{\alpha\beta})_\hbar, \\ \partial_0 \mathbf{\Theta} &= 1. \end{aligned} \quad (3.39)$$

The $\mathbb{K}[[t_H]]$ -algebra on $H \otimes \mathbb{K}[[t_H]]$ defined by the set $\{A_{\alpha\beta}{}^\gamma\}$ in our main theorem can be, using the *graded symmetry*, identified with a sequences $\underline{m} = m_2, m_3, \dots$ of multilinear maps $m_n : S^n H \longrightarrow H$, of ghost number 0, defined by

$$m_n(e_{a_1}, \dots, e_{a_n}) = m_{a_1 \dots a_n}{}^\gamma e_\gamma.$$

for $n = 2, 3, \dots$, where $S^n H$ denote the graded symmetric products of H and

$$A_{\alpha\beta}{}^\gamma = m_{\alpha\beta}{}^\gamma + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} t^{\rho_\ell} \dots t^{\rho_1} m_{\rho_1 \dots \rho_\ell \alpha\beta}{}^\gamma.$$

The sequence of $\underline{m} = m_2, m_3, \dots$ of multi-linear maps satisfy the infinite set of relations summarized by the *relation* $A_{\alpha\beta}{}^\rho A_{\rho\gamma}{}^\sigma = A_{\beta\gamma}{}^\rho A_{\alpha\rho}{}^\sigma$. From these data we build a triple

$$(H[[\hbar]], 0, \underline{m}),$$

where m_n is understood to be extended by \hbar -adic continuity to a $\mathbb{k}[[\hbar]]$ -multi-linear map $m_n : S^n(H[[\hbar]]) \rightarrow H[[\hbar]]$. We call the above triple a structure of *on-shell QFT algebra*⁵ with zero differential on H with trivial quantum descendant algebra:

$$(H[[\hbar]], \underline{0}).$$

Recall that \mathbf{O}_α is the image of e_α under the quantization map

$$\mathbf{f} : H \rightarrow \mathcal{C}[[\hbar]],$$

which is a morphism of QFT complexes $(H[[\hbar]], 0) \rightarrow (\mathcal{C}[[\hbar]], \mathbf{K})$, i.e., $\mathbf{K}\mathbf{f} = 0$ and $\mathbf{f}(e_\alpha) = \mathbf{O}_\alpha$ and $\mathbf{K}\mathbf{O}_\alpha = 0$. Now, we like to interpret \mathbf{f} as a quasi-isomorphism from the on-shell QFT algebra $(H[[\hbar]], 0, \underline{m})$ to the BV QFT algebra $(\mathcal{C}[[\hbar]], \mathbf{K}, \cdot)$ as QFT algebra up to homotopy.⁶

For the above purpose, consider the *quantum master equation* (3.39), a solution to which $\Theta = t^\alpha \mathbf{O}_\alpha + \frac{1}{2} t^\beta t^\alpha \mathbf{O}_{\alpha\beta} + \dots$ shall be relabeled as follows

$$\Theta = t^\alpha \phi_1(e_\alpha) + \frac{1}{2!} t^\beta t^\alpha \phi_2(e_\alpha, e_\beta) + \sum_{n=3}^{\infty} \frac{1}{n!} t^{\alpha_n} \dots t^{\alpha_1} \phi_n(e_{\alpha_1}, \dots, e_{\alpha_n}), \quad (3.40)$$

where $\phi_1 = \mathbf{f}$ and $\phi_n(e_{\alpha_1}, \dots, e_{\alpha_n}) = \mathbf{O}_{\alpha_1 \dots \alpha_n}$ such that $\phi_n = \phi_n + \hbar \phi_n^{(1)} + \hbar \phi_n^{(2)} + \dots$ is interpreted as a sequence of \mathbb{k} -multilinear maps, parametrized by \hbar , on $S^n H$ into \mathcal{C} with ghost number 0. We also denote $\Lambda_{\alpha\beta} = \sum_{n=2}^{\infty} \frac{1}{n!} t^{\tilde{\alpha}_n} \dots t^{\tilde{\alpha}_1} \lambda_n(e_{\alpha_1}, \dots, e_{\alpha_n})$, where $t^{\tilde{\alpha}} := (-1)^{|\alpha|} t^\alpha$, such that $\lambda_n = \lambda_n + \hbar \lambda_n^{(1)} + \hbar \lambda_n^{(2)} + \dots$ is interpreted as a sequence of \mathbb{k} -multilinear maps, parametrized by \hbar , on $S^n H$ into \mathcal{C} with ghost number -1 . Then

⁵ This is just a special class of on-shell QFT algebra and general definition of on shell QFT algebra will be give in the 4-th paper in this series.

⁶ We didn't define general QFT algebra and its morphism, so the above statement is not valid as mathematical notion yet. But the whole purpose of this series of papers is to reach to the "correct" definition of QFT algebra and its morphism, which contains all relevant physical information.

the equation (3.39) implies that \mathbf{f} induces the sequence $\underline{\phi} = \phi_1, \phi_2, \phi_3, \dots$ of $\mathbb{k}[[\hbar]]$ -multilinear as follows:

$$\begin{aligned}
 \hbar \phi_2(e_\alpha, e_\beta) &= \phi_1(e_\alpha) \cdot \phi_1(e_\beta) - \phi_1(m_2(e_\alpha, e_\beta)) - \mathbf{K} \lambda_2(e_\alpha, e_\beta), \\
 \hbar \phi_3(e_\alpha, e_\beta, e_\gamma) &= \phi_2(e_\alpha, e_\beta) \cdot \phi_1(e_\gamma) + (-1)^{|e_\alpha||e_\beta|} \phi_1(e_\beta) \cdot \phi_2(e_\alpha, e_\gamma) \\
 &\quad - \phi_2(e_\alpha, m_2(e_\beta, e_\gamma)) - \phi_1(m_3(e_\alpha, e_\beta, e_\gamma)) \\
 &\quad - \mathbf{K} \lambda_3(e_\alpha, e_\beta, e_\gamma) - \left(\phi_1(e_\alpha), \lambda_2(e_\beta, e_\gamma) \right)_\hbar, \\
 &\vdots
 \end{aligned} \tag{3.41}$$

We note that the sequence $\underline{\phi} = \phi_1, \phi_2, \phi_3, \dots$ is determined by \mathbf{f} , the product \cdot in \mathcal{C} and $\underline{m} = m_2, m_3, \dots$ in H up to homotopy. In particular $\hbar \phi_2$ measures the failure of $\mathbf{f} = \phi_1$ be an algebra map $(H[[\hbar]], m_2) \rightarrow (\mathcal{C}[[\hbar]], \cdot)$ up to homotopy. In other words, the first condition for the morphism \mathbf{f} of QFT complex being a morphism of QFT algebra is that the failure of \mathbf{f} from being an algebra map $(H[[\hbar]], m_2) \rightarrow (\mathcal{C}[[\hbar]], \cdot)$ up to homotopy must be divisible by \hbar . Similarly, the quantum master equation governs an elaborate sequence of divisibility conditions by \hbar^n of \mathbf{f} with respect to the product \cdot in $\mathcal{C}[[\hbar]]$ and $\underline{m} = m_2, m_3, \dots$ in $H[[\hbar]]$ up to homotopy.

We now recall that the (3.40) also solves quantum descendant equation:

$$\mathbf{K} \Theta + \frac{1}{2} (\Theta, \Theta)_\hbar = 0,$$

as an *automatic* consequence of quantum master equation. This means that the sequence $\underline{\phi} = \phi_1, \phi_2, \phi_3, \dots$ is a distinguished L_∞ -morphism from the trivial quantum descendant algebra $(H[[\hbar]], \underline{0})$ to the quantum descendant DGLA $(\mathcal{C}[[\hbar]], \mathbf{K}, (\cdot, \cdot)_\hbar)$ at the chain level, both regarded as L_∞ -algebras. So we call $\underline{\phi} = (\phi_1, \phi_2, \phi_3, \dots)$ quantum descendant morphism, or descendant morphism of quantum descendant algebra. It should be clear that not every morphism from $(H[[\hbar]], \underline{0})$ to $(\mathcal{C}[[\hbar]], \mathbf{K}, (\cdot, \cdot)_\hbar)$ as L_∞ -algebra over $\mathbb{k}[[\hbar]]$ is quantum descendant morphism.

It also follows that the classical limit $\underline{\phi} = \phi_1, \phi_2, \phi_3, \dots$ of $\underline{\phi} = \phi_1, \phi_2, \phi_3, \dots$ is a distinguished L_∞ -quasi-isomorphism from H to \mathcal{C} . Thus the moduli space \mathcal{M} defined by the MC equation of the DGLA $(\mathcal{C}, Q, (\cdot, \cdot))$ is smooth-formal. It should be clear that not every L_∞ quasi-isomorphism from H to \mathcal{C} is the classical limit of a quantum descendant morphism. Equivalently, not every versal solution to the MC equation of the DGLA $(\mathcal{C}, Q, (\cdot, \cdot))$ is originated from solution to the quantum master equation. We have called the versal solution obtained from the classical limit of quantum descendant of morphism of QFT algebra the quantum coordinates on \mathcal{M} .

Now we turn to the similar interpretation of the free energy \mathbf{F} . From (3.38), we deduce that

$$\hbar \varphi_{\alpha\beta} = \varphi_\alpha \cdot \varphi_\beta - m_{\alpha\beta}^\sigma \varphi_\sigma, \quad (3.42)$$

where $\varphi_{\alpha\beta} := \partial_\alpha \partial_\beta \mathbf{F} \big|_{t_H=0}$ and \cdot now denotes the multiplication of the ground field \mathbb{k} . The above equation is to be compared with the first relation in (3.41). We can interpret φ_α as the image of $e_\alpha \in H$ by a map $\varphi_1 : H \rightarrow \mathbb{k}[[\hbar]]$, i.e., $\varphi_1(e_\alpha) = \varphi_\alpha$, defined as $\varphi_1 = \tilde{\mathbf{t}} = \tilde{\mathbf{c}} \circ \mathbf{f}$, where $\tilde{\mathbf{c}}$ denote the normalized QFT cycle such that $\tilde{\mathbf{c}}(1) = 1$. Then (3.42) can be viewed as the definition of $\varphi_2(e_\alpha, e_\beta) = \varphi_{\alpha\beta}$:

$$\hbar \varphi_2(e_\alpha, e_\beta) := \varphi_1(e_\alpha) \cdot \varphi_1(e_\beta) - \varphi_1(m_2(e_\alpha, e_\beta)), \quad (3.43)$$

which measures the failure of $\varphi_1 = \tilde{\mathbf{t}}$ being an algebra map $(H[[\hbar]], m_2) \rightarrow (\mathbb{k}[[\hbar]], \cdot)$. In general, we have the following infinite sequence of relations (compare with (3.41))

$$\begin{aligned} \hbar \varphi_2(e_\alpha, e_\beta) &= \varphi_1(e_\alpha) \cdot \varphi_1(e_\beta) - \varphi_1(m_2(e_\alpha, e_\beta)), \\ \hbar \varphi_3(e_\alpha, e_\beta, e_\gamma) &= \varphi_2(e_\alpha, e_\beta) \cdot \varphi_1(e_\gamma) + (-1)^{|e_\alpha||e_\beta|} \varphi_1(e_\beta) \cdot \varphi_2(e_\alpha, e_\gamma) \\ &\quad - \varphi_2(e_\alpha, m_2(e_\beta, e_\gamma)) - \varphi_1(m_3(e_\alpha, e_\beta, e_\gamma)), \\ &\vdots \end{aligned} \quad (3.44)$$

which is formal power series expansion of the differential relations (3.38) at $t_H = 0$ such that $\partial_{\alpha_1} \cdots \partial_{\alpha_n} \mathbf{F} \big|_{t_H=0} = \varphi_n(e_{\alpha_1}, \dots, e_{\alpha_n})$, i.e., the formal expansion of \mathbf{F} at t_H is

$$\mathbf{F} = t^\alpha \varphi_1(e_\alpha) + \frac{1}{2!} t^\beta t^\alpha \varphi_2(e_\alpha, e_\beta) + \sum_{n=3}^{\infty} \frac{1}{n!} t^{\alpha_n} \cdots t^{\alpha_1} \varphi_n(e_{\alpha_1}, \dots, e_{\alpha_n}).$$

Note that the resulting recursive relations for the sequence $\underline{\varphi} = \varphi_1, \varphi_2, \dots$ are to be regarded as definitions of φ_n by $\tilde{\mathbf{t}} = \varphi_1$, and $\underline{m} = m_2, m_3, \dots$ in H and the ordinary multiplication of the ground field \mathbb{k} .

Now we can interpret the map $\tilde{\mathbf{t}} : H[[\hbar]] \rightarrow \mathbb{k}[[\hbar]]$ as a morphism of QFT algebra from the QFT algebra $(H[[\hbar]], 0, \underline{m})$ with zero differential to the trivial QFT algebra $(\mathbb{k}[[\hbar]], 0, \cdot)$ with zero differential. Both the QFT algebras have trivial quantum descendant algebra and the sequence $\underline{\varphi} = \varphi_1, \varphi_2, \dots$ is the quantum descendant morphism - a distinguished morphism on $H[[\hbar]]$ into $\mathbb{k}[[\hbar]]$ as L_∞ -algebras over $\mathbb{k}[[\hbar]]$.

The moral of the above story is that we have a confirmation of our metaphor that quantum field theory is a study of morphisms of QFT algebras, which contains rather complete information on quantum correlation functions.

4. QFT Integral and Quantum WDVV Equation

In this section we consider a class of BV QFT with an additional datum called a QFT integral. A QFT integral and QFT cycle have the same classical property and could be viewed as two different ways of quantization of classical cycle. A QFT integral is responsible for formal Frobenius manifold structure on moduli space in semi-classical case and gives certain quantum version of it in general.

4.1. QFT Integral

We begin with defining QFT integral and discuss motivations behind the notion afterward.

Definition 4.1. A QFT integral \oint of dimension N in BV QFT algebra is a sequence of \mathbb{k} -linear maps on \mathcal{C} into \mathbb{k} , parametrized by \hbar ,

$$\oint = \int + \hbar \int^{(1)} + \hbar^2 \int^{(2)} + \cdots : \mathcal{C} \rightarrow \mathbb{k}[[\hbar]],$$

of ghost number $-N$ satisfying

$$\oint \mathbf{K}a \cdot b = -(-1)^{|\mathbf{a}|} \oint a \cdot \mathbf{K}b,$$

for $\forall a, b \in \mathcal{C}$.

Remark 4.1. Note that QFT integral \oint also define a $\mathbb{k}[[\hbar]]$ -linear map on $\mathcal{C}[[\hbar]]$ into $\mathbb{k}[[\hbar]]$ such that $\oint \mathbf{K}\mathbf{a} \cdot \mathbf{b} = -(-1)^{|\mathbf{a}|} \oint \mathbf{a} \cdot \mathbf{K}\mathbf{b}$ for $\forall \mathbf{a}, \mathbf{b} \in \mathcal{C}[[\hbar]]$.

Corollary 4.1. A sequence \oint of \mathbb{k} -linear maps on \mathcal{C} into \mathbb{k} parametrized by \hbar is a QFT integral if and only if, for $\forall a, b \in \mathcal{C}$,

$$\begin{aligned} \oint \mathbf{K}a &= 0, \\ \oint (a, b)_{\hbar} &= 0, \end{aligned}$$

or equivalently $\oint \mathbf{K}\mathbf{a} = \oint (\mathbf{a}, \mathbf{b})_{\hbar} = 0$ for $\forall \mathbf{a}, \mathbf{b} \in \mathcal{C}[[\hbar]]$.

Proof. Set $b = 1$ in definition 4.1 and use the property that 1 is a center of the bracket to see that $\oint \mathbf{K}a = 0$ for $\forall a \in \mathcal{C}$. Applying \oint to the identity

$$\mathbf{K}(a \cdot b) = \mathbf{K}a \cdot b + (-1)^{|a|} a \cdot \mathbf{K}b - \hbar(-1)^{|a|} (a, b)_{\hbar}, \quad (4.1)$$

we have

$$0 = \oint \mathbf{K}a \cdot b + (-1)^{|a|} \int a \cdot \mathbf{K}b = \oint \mathbf{K}(a \cdot b) + \hbar(-1)^{|a|} \oint (a, b)_{\hbar}.$$

It follows that $\oint (a, b)_{\hbar} = 0$ for $\forall a, b \in \mathcal{C}$ since $\oint \mathbf{K}(a \cdot b) = 0$. Conversely assume that $\oint \mathbf{K}a = \oint (a, b)_{\hbar} = 0$ for $\forall a, b$. Then the identity (4.1) implies that $\oint \mathbf{K}a \cdot b + (-1)^{|a|} \int a \cdot \mathbf{K}b = 0$. \square

The above corollary implies that a QFT integral is automatically a QFT cycle, i.e., $\oint \mathbf{K} = 0$. But the converse is not true in general and QFT integral has the additional property that $\oint (a, b)_{\hbar} = 0$. A BV QFT algebra with a QFT integral also defines a BV QFT, since a QFT integral is a QFT cycle. In general we tend to regard a QFT integral an additional structure to a BV QFT.

Now consider the leading two relations for a QFT integral:

$$\begin{aligned} \int Qa \cdot b + (-1)^{|a|} \int a \cdot Qb &= 0. \\ \int K^{(1)}a \cdot b + (-1)^{|a|} \int a \cdot K^{(1)}b &= - \int^{(1)} Qa \cdot b - (-1)^{|a|} \int^{(1)} a \cdot Qb. \end{aligned} \quad (4.2)$$

We recall that the classical limit of BV QFT algebra $(\mathcal{C}[[\hbar]], \mathbf{K}, \cdot)$ is a CDGA (\mathcal{C}, Q, \cdot) . Then the 1st condition (4.2) means that the classical limit \int of \oint is a cycle of the CDGA (\mathcal{C}, Q, \cdot) , i.e., $\int \mathcal{C} \rightarrow \mathbb{k}$ and $\int Q = 0$. The condition $\int Q = 0$ is equivalent the first condition in (4.2), since $Q(a \cdot b) = Qa \cdot b + (-1)^{|a|} a \cdot Qb$. The classical limit $\mathfrak{c}^{(0)}$ of a QFT cycle $\mathfrak{c} = \mathfrak{c}^{(0)} + \hbar \mathfrak{c}^{(1)} + \dots$ also has the same property, while $\mathfrak{c}((a, b)_{\hbar}) \neq 0$ in general. We may view QFT cycle and QFT integral as two different ways of quantizing cycle of CDGA.

The classical limit \int of a QFT integral \oint induces a unique graded symmetric \mathbb{k} -bilinear pairing $\langle \cdot, \cdot \rangle : H^i \otimes H^{N-i} \rightarrow \mathbb{k}$ on the cohomology H of the cochain complex (\mathcal{C}, Q) defined by, for $x, y \in H$,

$$\langle x, y \rangle := \int f(x) \cdot f(y).$$

The pairing is a homotopy invariant since Q is a derivation of the product \cdot . We recall that there is a unique graded commutative and associative product $m_2 : H \otimes H \rightarrow H$ on H . Then

Lemma 4.1. *The triple $(H, m_2, \langle \cdot, \cdot \rangle)$ is a graded commutative Frobenius algebra, i.e., m_2 is a graded commutative and associative product and*

$$\langle x, y \rangle = (-1)^{|x||y|} \langle y, x \rangle, \quad \langle x, m_2(y, z) \rangle = \langle m_2(x, y), z \rangle$$

Proof. The graded commutativity of $\langle \cdot, \cdot \rangle$ follows from the graded commutativity of the product \cdot on \mathcal{C} . Recall that a \mathbb{k} -linear way of choosing representative of every element in H is a cochain map $f : (H, 0) \rightarrow (\mathcal{C}, Q)$, which induces the identity map on H and is a morphism of CDGA $f : (H, 0, m_2) \rightarrow (\mathcal{C}, Q, \cdot)$ up to homotopy: for $\forall x, y \in H$, $f(x) \cdot f(y) - f(m_2(x, y)) = Q\lambda_2(x, y)$ such that $f(x) \in \text{Ker } Q \cap \mathcal{C}$ and $[f(x)] = x$. Then, by definition,

$$\begin{aligned} \langle x, m_2(y, z) \rangle &:= \int f(x) \cdot f(m_2(y, z)) \\ &= \int f(x) \cdot (f(y) \cdot f(z) - Q\lambda_2(y, z)) \\ &= \int f(x) \cdot (f(y) \cdot f(z)), \end{aligned}$$

where we have used $\int (\text{Ker } Q) \cdot \text{Im } Q = 0$ for the last equality. By the similar manipulation we have

$$\begin{aligned} \langle m_2(x, y), z \rangle &:= \int f(m_2(x, y)) \cdot f(z) \\ &= \int (f(x) \cdot f(y)) \cdot f(z). \end{aligned}$$

Hence

$$\begin{aligned} \langle x, m_2(y, z) \rangle - \langle m_2(x, y), z \rangle &= \int (f(x) \cdot (f(y) \cdot f(z)) - (f(x) \cdot f(y)) \cdot f(z)) \\ &= 0, \end{aligned}$$

as is claimed. \square

Remark 4.2. The relation $\langle x, m_2(y, z) \rangle = \langle m_2(x, y), z \rangle$ together with the commutativity of m_2 is equivalent to the following relations

– left-invariance:

$$\langle x, m_2(y, z) \rangle = (-1)^{|x||y|} \langle y, m_2(x, z) \rangle,$$

– right-invariance:

$$\langle m_2(x, y), z \rangle = (-1)^{|y||z|} \langle m_2(x, z), y \rangle,$$

as well as to

– left-cyclic:

$$\langle x, m_2(y, z) \rangle = (-1)^{|x|(|y|+|z|)} \langle y, m_2(z, x) \rangle,$$

– right-cyclic:

$$\langle m_2(x, y), z \rangle = (-1)^{(|x|+|y|)|z|} \langle m_2(z, x), y \rangle,$$

Now we consider the quantum extension map $\mathbf{f} = f + \hbar f^{(1)} + \dots : H \rightarrow \mathcal{C}[[\hbar]]$ such that $\mathbf{K}\mathbf{f} = \mathbf{f}\mathbf{K}$. We may want to quantize the \mathbb{k} -bilinear pairing $\langle \cdot, \cdot \rangle : H^i \otimes H^{N-i} \rightarrow \mathbb{k}$ to certain well-defined sequence $\langle \cdot, \cdot \rangle_{\hbar} = \langle \cdot, \cdot \rangle + \hbar \langle \cdot, \cdot \rangle^{(1)} + \hbar^2 \langle \cdot, \cdot \rangle^{(2)} + \dots$ of \mathbb{k} -bilinear pairings parametrized by \hbar . It is natural to try

$$\langle x, y \rangle_{\hbar} = \int \mathbf{f}(x) \cdot \mathbf{f}(y),$$

which does not work, in general, since it depends on quantum homotopy. The first obstruction is \mathbf{K} , which is assumed to vanish in this paper. Hence $\mathbf{K}\mathbf{f} = 0$ and an arbitrary quantum extension map \mathbf{f}' homotopic to \mathbf{f} is given by $\mathbf{f}' = \mathbf{f} + \mathbf{K}\mathbf{s}$. Then

$$\begin{aligned} \mathbf{f}'(x) \cdot \mathbf{f}'(y) - \mathbf{f}(x) \cdot \mathbf{f}(y) &= \mathbf{f}(x) \cdot \mathbf{K}\mathbf{s}(y) + \mathbf{K}\mathbf{s}(x) \cdot \mathbf{f}(y) + \mathbf{K}\mathbf{s}(x) \cdot \mathbf{K}\mathbf{s}(y) \\ &= (-1)^{|x|} \mathbf{K}(\mathbf{f}(x) \cdot \mathbf{s}(y)) + \mathbf{K}(\mathbf{s}(x) \cdot \mathbf{f}(y)) \\ &\quad + \hbar(\mathbf{f}(x), \mathbf{s}(y))_{\hbar} + \hbar(-1)^{|x|-1}(\mathbf{s}(x), \mathbf{f}(y))_{\hbar}. \end{aligned}$$

The above computation implies that the integral \int must be quantized to a QFT integral $\oint = \int + \hbar \int^{(1)} + \dots$ such that $\int \mathbf{f}(x) \cdot \mathbf{f}(y)$ is quantum homotopy invariant.

Remark 4.3. In some case \int , satisfying $\int Qa \cdot b = -(-1)^{|a|} \int a \cdot Qb$, may be itself be a QFT integral. We call such a QFT integral $\int \equiv \oint$ semi-classical and note that it has to satisfy

$$\int K^{(n)}a \cdot b = -(-1)^{|a|} \int a \cdot K^{(n)}b,$$

for all $n = 1, 2, 3, \dots$. By the way the confusing expression that "something is semi-classical" really means that "something classical which is also quantum by itself".

Definition 4.2. (Lemma) Let \oint be a QFT integral of an anomaly-free BV QFT algebra, that is $\mathbf{K} = 0$ on H . Then there is a sequence of $\langle \cdot, \cdot \rangle_{\hbar} = \langle \cdot, \cdot \rangle + \hbar \langle \cdot, \cdot \rangle^{(1)} + \hbar^2 \langle \cdot, \cdot \rangle^{(2)} + \dots$ of \mathbb{k} -bilinear pairings parametrized by \hbar , $\langle \cdot, \cdot \rangle_{\hbar} : H \otimes H \rightarrow \mathbb{k}[[\hbar]]$, defined by

$$\langle x, y \rangle_{\hbar} = \oint \mathbf{f}(x) \cdot \mathbf{f}(y),$$

which is a quantum homotopy invariant.

Definition 4.3. A QFT integral \oint is called *non-degenerate* if the \mathbb{k} -bilinear pairing on the cohomology H defined by its classical limit \int is non-degenerate.

Proposition 4.1.

$$\begin{aligned} \langle x, m_2(y, z) \rangle_{\hbar} - \langle m_2(x, y), z \rangle_{\hbar} &= -\hbar \oint \phi_1(x) \cdot \phi_2(y, z) - \phi_2(x, y) \cdot \phi_1(z), \\ \langle x, m_2(y, z) \rangle_{\hbar} - (-1)^{|x||y|} \langle y, m_2(x, z) \rangle_{\hbar} &= -\hbar \oint \phi_1(x) \cdot \phi_2(y, z) - (-1)^{|x||y|} \phi_1(y) \cdot \phi_2(x, z), \\ \langle m_2(x, y), z \rangle_{\hbar} - (-1)^{|y||z|} \langle m_2(x, z), y \rangle_{\hbar} &= -\hbar \oint \phi_2(x, y) \cdot \phi_1(z) - (-1)^{|y||z|} \phi_2(x, z) \cdot \phi_1(y). \end{aligned}$$

Proof. Consider the relation

$$\hbar \phi_2(x, y) = \mathbf{f}(x) \cdot \mathbf{f}(y) - \mathbf{f}(m_2(x, y)) - \mathbf{K} \lambda_2(x, y).$$

Then, by definition, we have

$$\begin{aligned} \langle x, m_2(y, z) \rangle_{\hbar} &:= \oint \mathbf{f}(x) \cdot \mathbf{f}(m_2(y, z)) \\ &= \oint \mathbf{f}(x) \cdot (\mathbf{f}(y) \cdot \mathbf{f}(z) - \mathbf{K} \lambda_2(y, z) - \hbar \phi_2(y, z)) \\ &= \oint \mathbf{f}(x) \cdot (\mathbf{f}(y) \cdot \mathbf{f}(z)) - \hbar \oint \phi_1(x) \cdot \phi_2(y, z) \end{aligned}$$

where we have used $\oint \text{Ker } \mathbf{K} \cdot \text{Im } \mathbf{K} = 0$ for the last equality. After the similar manipulation we also obtain that

$$\begin{aligned} \langle m_2(x, y), z \rangle_{\hbar} &:= \oint \mathbf{f}(m_2(x, y)) \cdot \mathbf{f}(z) \\ &= \oint (\mathbf{f}(x) \cdot \mathbf{f}(y)) \cdot \mathbf{f}(z) - \hbar \oint \phi_2(x, y) \cdot \phi_1(z). \end{aligned}$$

Now the associativity of the product \cdot in \mathcal{C} implies that

$$\langle x, m_2(y, z) \rangle_{\hbar} - \langle m_2(x, y), z \rangle_{\hbar} = -\hbar \oint (\phi_1(x) \cdot \phi_2(y, z) - \phi_2(x, y) \cdot \phi_1(z)).$$

The remaining two relations can be proved similarly. \square

Definition 4.4. (Lemma) Let \oint be a QFT integral of an anomaly-free BV QFT algebra, that is $\kappa = 0$ on H . Then there is a sequence $\langle *, *, * \rangle_{\hbar} = \langle *, *, * \rangle + \hbar \langle *, *, * \rangle^{(1)} + \hbar^2 \langle *, *, * \rangle^{(2)} \dots$ of \mathbb{K} -trilinear parings parametrized by \hbar :

$$\langle *, *, * \rangle_{\hbar} : H \otimes H \otimes H \rightarrow \mathbb{K}[[\hbar]],$$

which is a quantum homotopy invariant, where

$$\langle x, y, z \rangle_{\hbar} := \oint \left(\phi_2(x, y) \cdot \phi_1(z) + (-1)^{|x||y|} \phi_1(y) \cdot \phi_2(x, z) \right).$$

such that

$$(1) \langle x, y, z \rangle_{\hbar} = (-1)^{|y||z|} \langle x, z, y \rangle_{\hbar},$$

$$(2) \langle e, y, z \rangle_{\hbar} = 0,$$

$$(3) \langle x, m_2(y, z) \rangle_{\hbar} - \langle m_2(x, y), z \rangle_{\hbar} = \hbar \left(\langle x, y, z \rangle_{\hbar} - (-1)^{|x||z|+|y||z|} \langle z, x, y \rangle_{\hbar} \right)$$

$$(4) \langle x, m_2(y, z) \rangle_{\hbar} - (-1)^{|x||y|} \langle y, m_2(x, z) \rangle_{\hbar} = \hbar \left(\langle x, y, z \rangle_{\hbar} - (-1)^{|x||y|} \langle y, x, z \rangle_{\hbar} \right)$$

$$(5) \langle m_2(x, y), z \rangle_{\hbar} - (-1)^{|y||z|} \langle m_2(x, z), y \rangle_{\hbar} = \hbar \left((-1)^{|x||z|+|y||z|} \langle z, x, y \rangle_{\hbar} - (-1)^{|x||y|} \langle y, x, z \rangle_{\hbar} \right)$$

Proof. Exercise.

We may go on to define higher pairings and examine their properties. Instead we look for generating formula using the solution Θ of quantum master equation in the case that H is finite dimensional for each ghost number. Let $\{e_\alpha\}$ be the basis of H introduced in the beginning of section 3. Then

$$\begin{aligned} \langle e_\alpha, e_\beta \rangle_{\hbar} &= \oint \mathbf{O}_\alpha \cdot \mathbf{O}_\beta, \\ \langle e_\alpha, e_\beta, e_\gamma \rangle_{\hbar} &= \oint \left(\mathbf{O}_{\alpha\beta} \cdot \mathbf{O}_\gamma + (-1)^{|\alpha||\beta|} \mathbf{O}_\beta \cdot \mathbf{O}_{\alpha\gamma} \right). \end{aligned}$$

Define

$$\begin{aligned} \mathbf{g}_{\beta\gamma} &:= \oint \partial_\beta \Theta \cdot \partial_\gamma \Theta \\ &= \oint \mathbf{O}_\alpha \cdot \mathbf{O}_\beta + t^\alpha \oint \left(\mathbf{O}_{\alpha\beta} \cdot \mathbf{O}_\gamma + (-1)^{|\alpha||\beta|} \mathbf{O}_\beta \cdot \mathbf{O}_{\alpha\gamma} \right) + \dots \end{aligned}$$

such that

$$\mathbf{g}_{\beta\gamma} = \langle e_\beta, e_\gamma \rangle_{\hbar} + t^\alpha \langle e_\alpha, e_\beta, e_\gamma \rangle_{\hbar} + \dots$$

We then define n -nary pairing on $H^{\otimes n}$ into $\mathbb{k}[[\hbar]]$ by

$$\langle e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_{n-2}}, e_{\beta}, e_{\gamma} \rangle_{\hbar} := \partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_{n-2}} \mathbf{g}_{\beta\gamma} \Big|_{t_H=0}.$$

Proposition 4.2. *Let $\mathbf{g}_{\beta\gamma} := \oint \partial_{\beta} \Theta \cdot \partial_{\gamma} \Theta \in \mathbb{k}[[t_H]][[\hbar]]$. Then, the following relations are satisfied:*

- (1) $\mathbf{g}_{\beta\gamma} = (-1)^{|\beta||\gamma|} \mathbf{g}_{\gamma\beta}$,
- (2) $\partial_0 \mathbf{g}_{\beta\gamma} = 0$,
- (3) $\hbar \left(\partial_{\alpha} \mathbf{g}_{\beta\gamma} - (-1)^{|\alpha||\beta|} \partial_{\beta} \mathbf{g}_{\alpha\gamma} \right) = A_{\beta\gamma}{}^{\rho} \mathbf{g}_{\alpha\rho} - (-1)^{|\alpha||\beta|} A_{\alpha\gamma}{}^{\rho} \mathbf{g}_{\beta\rho}$,
- (4) $\partial_{\alpha} \mathbf{g}_{\beta 0} - (-1)^{|\alpha||\beta|} \partial_{\beta} \mathbf{g}_{\alpha 0} = 0$.

Proof. The 1st property is obvious. The 2nd property follows from the quantum unity that $\partial_0 \Theta = 1$, which implies that $\partial_0 \partial_{\alpha} \Theta = 0$. For the 3rd property, use the quantum master equation $\partial_{\beta} \Theta \cdot \partial_{\gamma} \Theta = \hbar \partial_{\beta} \partial_{\gamma} \Theta + A_{\beta\gamma}{}^{\rho} \partial_{\rho} \Theta + \mathbf{K}_{\Theta} \Lambda_{\beta\gamma}$, which implies that

$$\begin{aligned} \mathbf{g}_{\beta\gamma} &= \hbar \oint \partial_{\beta} \partial_{\gamma} \Theta + A_{\beta\gamma}{}^{\rho} \oint \partial_{\rho} \Theta + \oint \mathbf{K} \Lambda_{\beta\gamma} + \oint (\Theta, \Lambda_{\beta\gamma})_{\hbar} \\ &= \hbar \oint \partial_{\beta} \partial_{\gamma} \Theta + A_{\beta\gamma}{}^{\rho} \oint \partial_{\rho} \Theta. \end{aligned}$$

Applying $\hbar \partial_{\alpha}$ to the above we have

$$\begin{aligned} \hbar \partial_{\alpha} \mathbf{g}_{\beta\gamma} &= \hbar^2 \oint \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \Theta + \hbar \partial_{\alpha} A_{\beta\gamma}{}^{\rho} \oint \partial_{\rho} \Theta + \hbar A_{\beta\gamma}{}^{\rho} \oint \partial_{\alpha} \partial_{\rho} \Theta \\ &= \hbar^2 \oint \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \Theta + \hbar \partial_{\alpha} A_{\beta\gamma}{}^{\rho} \oint \partial_{\rho} \Theta - A_{\beta\gamma}{}^{\rho} A_{\alpha\rho}{}^{\sigma} \oint \partial_{\sigma} \Theta \\ &\quad + A_{\beta\gamma}{}^{\rho} \mathbf{g}_{\alpha\rho}. \end{aligned}$$

Then the 3rd proposition follows from the potentiality and the associativity of $A_{\alpha\beta}{}^{\gamma}$. For the 4th property, set $\gamma = 0$ to relation (3) and use $A_{\beta 0}{}^{\rho} = \delta_{\beta}{}^{\rho}$. \square

4.2. WDVV equation as a semi-classical phenomena

We call a solution Θ to the quantum master equation semi-classical if Θ does not depend on \hbar , i.e., $\Theta = \Theta$. Then the quantum master equation is decomposed into the

following form

$$\begin{aligned}
\Theta_\alpha \cdot \Theta_\beta &= A_{\alpha\beta}{}^\gamma \Theta_\gamma + Q \Lambda_{\alpha\beta}^{(0)} + \left(\Theta, \Lambda_{\alpha\beta}^{(0)} \right)^{(0)}, \\
\Theta_{\alpha\beta} &= -K^{(1)} \Lambda_{\alpha\beta}^{(0)} - Q \Lambda_{\alpha\beta}^{(1)} - \left(\Theta, \Lambda_{\alpha\beta}^{(1)} \right)^{(0)} - \left(\Theta, \Lambda_{\alpha\beta}^{(0)} \right)^{(1)}, \\
0 &= -\sum_{j=1}^n K^{(j)} \Lambda_{\alpha\beta}^{(n-j)} - Q \Lambda_{\alpha\beta}^{(n)} - \sum_{j=0}^n \left(\Theta, \Lambda_{\alpha\beta}^{(j)} \right)^{(n-j)} \quad \text{for } n \geq 2,
\end{aligned} \tag{4.3}$$

where $\Theta_\alpha := \partial_\alpha \Theta$ and $\Theta_{\alpha\beta} := \partial_\alpha \partial_\beta \Theta$. The quantum descendant equation is, then, decomposed as follows

$$\begin{aligned}
Q\Theta + \frac{1}{2} (\Theta, \Theta)^{(0)} &= 0, \\
K^{(1)}\Theta + \frac{1}{2} (\Theta, \Theta)^{(1)} &= 0, \\
K^{(n)}\Theta + \frac{1}{2} (\Theta, \Theta)^{(n)} &= 0 \quad \text{for } n \geq 2.
\end{aligned} \tag{4.4}$$

We say a BV QFT algebra $(\mathcal{C}[[\hbar]], \mathbf{K} = Q + \hbar K^{(1)} + \hbar^2 K^{(2)} + \dots, \cdot)$ is semi-classical if it admit a semi-classical solution to the quantum master equation. We say a QFT integral \oint semi-classical if $\oint = \int$.

Lemma 4.2. *For a semi-classical BV QFT algebra with a semi-classical QFT integral $\oint = \int$, the metric $\mathbf{g}_{\alpha\beta} = \oint \Theta_\alpha \cdot \Theta_\beta$ satisfies the following properties*

1. *it does not depend on \hbar : $\mathbf{g}_{\alpha\beta} = g_{\alpha\beta}$,*
2. *it is flat: $\partial_\alpha g_{\beta\gamma} = 0$*
3. *it is compatible with the 3-tensor $A_{\alpha\beta}{}^\gamma: A_{\alpha\beta}{}^\rho g_{\rho\gamma} = A_{\beta\gamma}{}^\rho g_{\alpha\rho}$.*

Proof. Note that $\oint Qa \cdot b = -(-1)^{|a|} \oint a \cdot Qb$ and $\oint K^{(n)}a \cdot b = -(-1)^{|a|} \oint a \cdot K^{(n)}b$ for all $a, b \in \mathcal{C}$ and for all $n \geq 1$, since \oint is semiclassical. It also that $\oint(\cdot, \cdot)^{(\ell)} = 0$ for all $\ell = 0, 1, 2, \dots$ due to corollary 4.1 in the semi-classical case.

1. Property 1 is obvious since both \oint and Θ_α do not depend on \hbar .

2. By definition, we have

$$\partial_\alpha g_{\beta\gamma} = \oint \Theta_{\alpha\beta} \cdot \Theta_\gamma + (-1)^{|\alpha||\beta|} \oint \Theta_\alpha \cdot \Theta_{\beta\gamma}.$$

Consider $\oint \Theta_{\alpha\beta} \cdot \Theta_\gamma$ and substitute $\Theta_{\alpha\beta}$ using the quantum master equation (4.3) to have

$$\begin{aligned} \oint \Theta_{\alpha\beta} \cdot \Theta_\gamma &= - \oint \left(K^{(1)} \Lambda_{\alpha\beta}^{(0)} \cdot \Theta_\gamma + \left(\Theta, \Lambda_{\alpha\beta}^{(0)} \right)^{(1)} \cdot \Theta_\gamma \right) \\ &\quad - \oint \left(Q \Lambda_{\alpha\beta}^{(1)} \cdot \Theta_\gamma + \left(\Theta, \Lambda_{\alpha\beta}^{(1)} \right)^{(0)} \cdot \Theta_\gamma \right). \end{aligned}$$

Consider two terms in the RHS above separately;

-The 1st term: We have

$$\oint K^{(1)} \Lambda_{\alpha\beta}^{(0)} \cdot \Theta_\gamma = (-1)^{|\alpha|+|\beta|} \oint \Lambda_{\alpha\beta}^{(0)} \cdot K^{(1)} \Theta_\gamma = -(-1)^{|\alpha|+|\beta|} \oint \Lambda_{\alpha\beta}^{(0)} \cdot \left(\Theta, \Theta_\gamma \right)^{(1)},$$

where we have used $\oint K^{(1)} a \cdot b = -(-1)^{|a|} \oint a \cdot K^{(1)} b$ for the 1st equality (note that $|\Lambda_{\alpha\beta}^{(\ell)}| = |e_\alpha| + |e_\beta| - 1$ for all $\ell = 0, 1, 2, \dots$) and $K^{(1)} \Theta_\gamma + (\Theta, \Theta_\gamma)^{(1)} = 0$, which is a consequence of the quantum descendant equation (4.4), for the 2nd equality. Hence, we have

$$\begin{aligned} \oint \left(K^{(1)} \Lambda_{\alpha\beta}^{(0)} \cdot \Theta_\gamma + \left(\Theta, \Lambda_{\alpha\beta}^{(0)} \right)^{(1)} \cdot \Theta_\gamma \right) &= \oint \left(-(-1)^{|\alpha|+|\beta|} \Lambda_{\alpha\beta}^{(0)} \cdot \left(\Theta, \Theta_\gamma \right)^{(1)} + \left(\Theta, \Lambda_{\alpha\beta}^{(0)} \right)^{(1)} \cdot \Theta_\gamma \right) \\ &= \oint \left(\Theta, \Lambda_{\alpha\beta}^{(0)} \cdot \Theta_\gamma \right)^{(1)} = 0, \end{aligned}$$

where we have used the Poisson-law (2.2) for the 2nd equality and the property that $\oint (,)^{(\ell)} = 0$ for the last equality.

-The 2nd term: After the similar manipulations using $\oint Q a \cdot b = -(-1)^{|a|} \oint a \cdot Q b$ and $Q \Theta_\gamma + (\Theta, \Theta_\gamma)^{(0)} = 0$, we have

$$\oint Q \Lambda_{\alpha\beta}^{(1)} \cdot \Theta_\gamma = (-1)^{|\alpha|+|\beta|} \oint \Lambda_{\alpha\beta}^{(1)} \cdot Q \Theta_\gamma = -(-1)^{|\alpha|+|\beta|} \oint \Lambda_{\alpha\beta}^{(1)} \cdot \left(\Theta, \Theta_\gamma \right)^{(0)},$$

such that

$$\begin{aligned} \oint \left(Q \Lambda_{\alpha\beta}^{(1)} \cdot \Theta_\gamma + \left(\Theta, \Lambda_{\alpha\beta}^{(1)} \right)^{(0)} \cdot \Theta_\gamma \right) &= \oint \left(-(-1)^{|\alpha|+|\beta|} \Lambda_{\alpha\beta}^{(1)} \cdot \left(\Theta, \Theta_\gamma \right)^{(0)} + \left(\Theta, \Lambda_{\alpha\beta}^{(1)} \right)^{(0)} \cdot \Theta_\gamma \right) \\ &= \oint \left(\Theta, \Lambda_{\alpha\beta}^{(1)} \cdot \Theta_\gamma \right)^{(0)} = 0. \end{aligned}$$

where we have used the Poisson-law (2.2) for the 2nd equality and the property that $\oint (,)^{(\ell)} = 0$ for the last equality.

Thus $\oint \Theta_{\alpha\beta} \cdot \Theta_\gamma = 0$, which also implies that $\oint \Theta_\alpha \cdot \Theta_{\beta\gamma} = 0$. Consequently we have $\partial_\alpha g_{\beta\gamma} = 0$.

(3) From the associativity of the product \cdot we have

$$\oint (\Theta_\alpha \cdot \Theta_\beta) \cdot \Theta_\gamma = \oint \Theta_\alpha \cdot (\Theta_\beta \cdot \Theta_\gamma). \quad (4.5)$$

The LHS of the above, after using the quantum master equation (4.3), becomes

$$\oint (\Theta_\alpha \cdot \Theta_\beta) \cdot \Theta_\gamma = \oint A_{\alpha\beta}{}^\gamma \Theta_\rho \cdot \Theta_\gamma + \oint \left(Q\Lambda_{\alpha\beta}^{(0)} \cdot \Theta_\gamma + (\Theta, \Lambda_{\alpha\beta}^{(0)})^{(0)} \cdot \Theta_\gamma \right).$$

Then the first term gives

$$\oint A_{\alpha\beta}{}^\gamma \Theta_\rho \cdot \Theta_\gamma = A_{\alpha\beta}{}^\gamma \oint \Theta_\rho \cdot \Theta_\gamma = A_{\alpha\beta}{}^\gamma g_{\rho\gamma},$$

while the second term vanishes, since

$$\begin{aligned} \oint \left(Q\Lambda_{\alpha\beta}^{(0)} \cdot \Theta_\gamma + (\Theta, \Lambda_{\alpha\beta}^{(0)})^{(0)} \cdot \Theta_\gamma \right) &= \oint \left((-1)^{|\alpha|+|\beta|} \Lambda_{\alpha\beta}^{(0)} \cdot Q\Theta_\gamma + (\Theta, \Lambda_{\alpha\beta}^{(0)})^{(0)} \cdot \Theta_\gamma \right) \\ &= \oint \left(-(-1)^{|\alpha|+|\beta|} \Lambda_{\alpha\beta}^{(0)} \cdot (\Theta, \Theta_\gamma)^{(0)} + (\Theta, \Lambda_{\alpha\beta}^{(0)})^{(0)} \cdot \Theta_\gamma \right) \\ &= \oint (\Theta, \Lambda_{\alpha\beta}^{(0)} \cdot \Theta_\gamma)^{(0)} = 0. \end{aligned}$$

Hence $\oint (\Theta_\alpha \cdot \Theta_\beta) \cdot \Theta_\gamma = A_{\alpha\beta}{}^\gamma g_{\rho\gamma}$. The similar computations for the RHS of (4.5) gives $\oint \Theta_\alpha \cdot (\Theta_\beta \cdot \Theta_\gamma) = A_{\beta\gamma}{}^\rho g_{\alpha\rho}$. It follows that

$$A_{\alpha\beta}{}^\gamma g_{\rho\gamma} = A_{\beta\gamma}{}^\rho g_{\alpha\rho}$$

as was claimed. \square

Corollary 4.2. *Let $A_{\alpha\beta\gamma} := A_{\beta\gamma}{}^\rho g_{\alpha\rho} \in \mathbb{k}[[t_H]]$. Then There exist $\Phi \in \mathbb{k}[[t_H]]$ such that*

$$\begin{aligned} A_{\alpha\beta\gamma} &= \frac{\partial^3 \Phi}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \\ g_{\beta\gamma} &= \frac{\partial^2 \Phi}{\partial t^0 \partial t^\beta \partial t^\gamma}. \end{aligned}$$

Proof. Standard.

Hence we just have established that a semi-classical BV QFT algebra with a semi-classical QFT integral induce a structure of Frobenius manifold on \mathcal{M} , albeit the flat metric $g_{\alpha\beta}$ may not be invertible. We have invertible metric if the semi-classical QFT integral is non-degenerate, leading to WDDV equation.

Example 4.1. Consider a BV QFT algebra $(\mathcal{C}[[\hbar]], \mathbf{K}, \cdot)$ with the property that $\mathbf{K} = Q - \hbar\Delta$, i.e., $K^{(1)} = -\Delta$ and $K^{(n)} = 0$ for all $n > 1$. Then the quadruples $(\mathcal{C}, Q, \Delta, \cdot)$ is a differential BV algebra. Assume that each cohomology class e_α of the complex (\mathcal{C}, Q) has a representative $f([e_\alpha]) = O_\alpha$ satisfying $\Delta O_\alpha = 0$. Hence $\mathbf{f}(e_\alpha) = f(e_\alpha)$ satisfying $\mathbf{K}f = 0$. Solving the quantum master equation order by order in t_H , it is obvious that $\Theta = \theta$. Then the quantum master equation is decomposed into the following form

$$\begin{aligned}\Theta_\alpha \cdot \Theta_\beta &= A_{\alpha\beta}{}^\gamma \Theta_\gamma + Q\Lambda_{\alpha\beta} + (\Theta, \Lambda_{\alpha\beta}), \\ -\hbar\Theta_{\alpha\beta} &= -\hbar\Delta\Lambda_{\alpha\beta}\end{aligned}$$

It follows that our special solution Θ has the form

$$\Theta = t^\alpha O_\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} t^{\alpha_n} \dots t^{\alpha_1} \Delta \lambda_{\alpha_1 \dots \alpha_n},$$

such that $\Delta\Theta = Q\Theta + (\Theta, \Theta) = 0$. The semi-classical QFT integral \oint satisfies $\oint Qa \cdot b = -(-1)^{|a|} \oint a \cdot Qb$ and $\oint \Delta a \cdot b = -(-1)^{|a|} \oint a \cdot \Delta b$. Hence corresponds to BK integral.⁷ The above is exactly the special versal solution in [11] corresponding to the flat coordinates on moduli space.

For some semi-classical BV QFT algebra, a semi-classical QFT integral may plays the role of QFT cycle. Then

Corollary 4.3. *For a BV QFT which underlying BV QFT algebra is semi-classical with QFT cycle, which is a semi-classical QFT integral, the set of 2-point quantum correlation functions is the flat metric $g_{\alpha\beta}$ and the set of 3-point quantum correlation functions is the 3-tensor $A_{\alpha\beta\gamma}$.*

Proof. By the assumption $\langle \Pi_{\alpha\beta} \rangle = \oint \Pi_{\alpha\beta}$, where $\Pi_{\alpha\beta} = \Theta_\alpha \cdot \Theta_\beta - \hbar\Theta_{\alpha\beta}$. From the quantum master equation (4.3) we have $\oint \Theta_{\alpha\beta} = 0$. Hence

$$\langle \Pi_{\alpha\beta} \rangle = \oint \Theta_\alpha \cdot \Theta_\beta = g_{\alpha\beta}.$$

Now consider the 3-point quantum correlators

$$\Pi_{\alpha\beta\gamma} = \Theta_\alpha \cdot \Theta_\beta \cdot \Theta_\gamma - \hbar\Theta_{\alpha\beta} \cdot \Theta_\gamma - \hbar\Theta_\alpha \cdot \Theta_{\beta\gamma} - \hbar(-1)^{|\alpha||\beta|} \Theta_\beta \cdot \Theta_{\alpha\gamma} + \hbar^2\Theta_{\alpha\beta\gamma},$$

where $\Theta_{\alpha\beta\gamma} = \partial_\alpha \Theta_{\beta\gamma}$. Then

$$\langle \Pi_{\alpha\beta\gamma} \rangle = \oint \Theta_\alpha \cdot \Theta_\beta \cdot \Theta_\gamma = A_{\alpha\beta\gamma},$$

⁷ Note that the ghost number for Δ operation is different from the convention in [11].

since we already have shown that $\oint \Theta_{\alpha\beta} \cdot \Theta_\gamma = \oint \Theta_\alpha \cdot \Theta_{\beta\gamma} = \oint \Theta_\beta \cdot \Theta_{\alpha\gamma} = 0$ while proving lemma 4.2 and $\oint \Theta_{\alpha\beta\gamma} = \partial_\alpha \oint \Theta_{\beta\gamma} = 0$. \square .

5. Proof of the Main Theorem

The purpose of this section is to prove the main theorem [theorem 3.1].

Let $\Theta^{[1]} := t^\alpha \mathbf{f}(e_\alpha) = t^\alpha \mathbf{O}_\alpha$ such that $\mathbf{K}\Theta^{[1]} = 0$, since $\mathbf{K}\mathbf{O}_\alpha = 0$. We set $\mathbf{P}(1) = \{\Theta^{[1]}, 0, 0\}$. For a natural number $n \geq 2$, we shall build an inductive system $\mathbf{P}(n)$

$$\mathbf{P}(1) \subset \mathbf{P}(2) \subset \mathbf{P}(3) \cdots \subset \mathbf{P}(n-1) \subset \mathbf{P}(n),$$

such that $n \rightarrow \infty$ limit implies theorem 3.1.

5.1. Setting up $\mathbf{P}(n)$

Definition 5.1. $\mathbf{P}(n)$, for a fixed $n \geq 2$, is a system consist of

$$\begin{aligned} \Theta &= \Theta^{[1]} + \Theta^{[2]} + \cdots + \Theta^{[n]}, \\ \Lambda_{a_2 a_1} &= \Lambda_{a_2 a_1}^{[0]} + \Lambda_{a_2 a_1}^{[1]} + \cdots + \Lambda_{a_2 a_1}^{[n-2]}, \\ A_{a_2 a_1}^\gamma &= A_{a_2 a_1}^{[0]\gamma} + A_{a_2 a_1}^{[1]\gamma} + \cdots + A_{a_2 a_1}^{[n-2]\gamma}, \end{aligned}$$

where, for $1 \leq k \leq n$, $0 \leq j \leq n-2$ and $0 \leq \ell \leq n-3$,

$$\begin{aligned} \Theta^{[k]} &= \frac{1}{k!} t^{\alpha_1} \cdots t^{\alpha_k} \mathbf{O}_{\alpha_k \cdots \alpha_1} \text{ where } \mathbf{O}_{\alpha_k \cdots \alpha_1} \in \mathcal{C}[[\hbar]]^{|\alpha_1| + \cdots + |\alpha_k|}, \\ \Lambda_{a_2 a_1}^{[j]} &= \frac{1}{j!} \bar{t}^{\rho_1} \cdots \bar{t}^{\rho_j} \lambda_{\rho_j \cdots \rho_1 a_2 a_1} \text{ where } \lambda_{\rho_j \cdots \rho_1 a_2 a_1} \in \mathcal{C}[[\hbar]]^{|\alpha_1| + \cdots + |\alpha_j| - 1}, \\ A_{a_2 a_1}^{[j]\gamma} &= \frac{1}{j!} t^{\rho_1} \cdots t^{\rho_j} m_{\rho_j \cdots \rho_1 a_2 a_1}^\gamma \text{ where } m_{\rho_j \cdots \rho_1 a_2 a_1}^\gamma \in \mathbb{K}, \end{aligned}$$

which satisfy the following properties:

1. *graded commutativity*: $A_{a_2 a_1}^\gamma - (-1)^{|\alpha_2||\alpha_1|} A_{a_1 a_2}^\gamma = 0 \text{ mod } t_H^{n-1}$,
2. *associativity*: $A_{a_3 a_2}^\rho A_{\rho a_1}^\gamma - A_{a_2 a_1}^\rho A_{a_3 \rho}^\gamma = 0 \text{ mod } t_H^{n-1}$,
3. *potentiality*: $\partial_{a_3} A_{a_2 a_1}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \partial_{a_2} A_{a_3 a_1}^\gamma = 0 \text{ mod } t_H^{n-2}$,
4. *unity* $A_{0\beta}^\gamma = 1$

5. quantum master equation:

$$\hbar \partial_{\alpha_2} \partial_{\alpha_1} \Theta = \partial_{\alpha_2} \Theta \cdot \partial_{\alpha_1} \Theta - A_{\alpha_2 \alpha_1}{}^\gamma \partial_\gamma \Theta - \mathbf{K} \Lambda_{\alpha_2 \alpha_1} - (\Theta, \Lambda_{\alpha_2 \alpha_1})_\hbar \mod t_H^{n-1},$$

6. quantum unity: $\partial_0 \Theta = 1 \mod t_H^n$.

7. quantum descendant equation: $\mathbf{K} \Theta + \frac{1}{2} (\Theta, \Theta)_\hbar = 0 \mod t_H^{n+1}$.

8. quantum gauge: $\Lambda_{\alpha_2 \alpha_1}$ satisfies

$$\begin{aligned} \Lambda_{\alpha_2 0} &= 0, \\ \Lambda_{\alpha_2 \alpha_1} &= (-1)^{|\alpha_2||\alpha_1|} \Lambda_{\alpha_1 \alpha_2}, \\ \hbar \partial_{\bar{\alpha}} \Lambda_{\beta \gamma} - (-1)^{|\alpha||\beta|} \hbar \partial_{\bar{\beta}} \Lambda_{\alpha \gamma} &= -(-1)^{|\alpha_3|} \partial_{\alpha_3} \Theta \cdot \Lambda_{\alpha_2 \alpha_1} + (-1)^{|\alpha_3||\alpha_2|+|\alpha_2|} \partial_{\alpha_2} \Theta \cdot \Lambda_{\alpha_3 \alpha_1} \\ &\quad - A_{\alpha_2 \alpha_1}{}^\rho \Lambda_{\alpha_3 \rho} + (-1)^{|\alpha_3||\alpha_2|} A_{\alpha_3 \alpha_1}{}^\rho \Lambda_{\alpha_2 \rho} \\ &\quad - B_{\alpha_3 \alpha_2 \alpha_1}{}^\rho \partial_\rho \Theta - \mathbf{K} X_{\alpha_3 \alpha_2 \alpha_1} - (\Theta, X_{\alpha_3 \alpha_2 \alpha_1})_\hbar \mod t_H^{n-2}, \end{aligned}$$

where $B_{\alpha_3 \alpha_2 \alpha_1}{}^\gamma \in \mathbb{K}[[t_H]] \mod t_H^{n-2}$ is a certain 4-tensor satisfying

$$\begin{aligned} B_{0 \alpha_2 \alpha_1}{}^\gamma &= B_{\alpha_3 0 \alpha_1}{}^\gamma = B_{\alpha_3 \alpha_2 0}{}^\gamma = 0 \mod t_H^{n-2}, \\ B_{\alpha_3 \alpha_2 \alpha_1}{}^\gamma + (-1)^{|\alpha_3||\alpha_2|} B_{\alpha_2 \alpha_3 \alpha_1}{}^\gamma &= 0 \mod t_H^{n-2}, \\ B_{\alpha_3 \alpha_2 \alpha_1} - (-1)^{|\alpha_2||\alpha_1|} B_{\alpha_3 \alpha_1 \alpha_2} + (-1)^{|\alpha_3|(|\alpha_2|+|\alpha_1|)} B_{\alpha_2 \alpha_1 \alpha_3} &= 0 \mod t_H^{n-2}, \end{aligned}$$

and $X_{\alpha_3 \alpha_2 \alpha_1} \in \mathcal{C}[[t_H]]^{|\alpha_3|+|\alpha_2|+|\alpha_1|-2} \mod t_H^n$ satisfying

$$\begin{aligned} X_{0 \alpha_2 \alpha_1} &= X_{\alpha_3 0 \alpha_1} = X_{\alpha_3 \alpha_2 0} = 0 \mod t_H^{n-2}, \\ X_{\alpha_3 \alpha_2 \alpha_1} + (-1)^{|\alpha_3||\alpha_2|} X_{\alpha_2 \alpha_3 \alpha_1} &= 0 \mod t_H^{n-2}, \\ X_{\alpha_3 \alpha_2 \alpha_1} - (-1)^{|\alpha_2||\alpha_1|} X_{\alpha_3 \alpha_1 \alpha_2} + (-1)^{|\alpha_3|(|\alpha_2|+|\alpha_1|)} X_{\alpha_2 \alpha_1 \alpha_3} &= 0 \mod t_H^{n-2}.. \end{aligned}$$

Remark that the 4-tensor $B_{\alpha \beta \gamma}{}^\rho$ does not contribute to $\mathbf{P}(2)$.

Note that we have already constructed $\mathbf{P}(2)$ as well as a sketch of constructing $\mathbf{P}(3)$ out of $\mathbf{P}(2)$ after the statement of theorem 3.1. We shall build $\mathbf{P}(n+1)$ out of assumed $\mathbf{P}(n)$.

Remark 5.1. The homogeneity $|e_\rho| t^\rho \partial_\rho A_{\alpha \beta}{}^\gamma = (|e_\gamma| - |e_\beta| - |e_\alpha|) A_{\alpha \beta}{}^\gamma$ follows from the quantum master equation by matching the ghost number.

5.1.1. *Some consequences of $\mathbf{P}(n)$.* Assuming $\mathbf{P}(n)$ we examine some of its consequences, which shall be used to build $\mathbf{P}(n+1)$ from $\mathbf{P}(n)$.

It is convenient to introduce the notation that $\Theta_\alpha := \partial_\alpha \Theta$ and $\Theta_{\beta\alpha} := \partial_\beta \Theta_\alpha$ such that the $\text{mod } t_H^{n-1}$ quantum master equation in $\mathbf{P}(n)$ is rewritten as follows

$$\hbar \Theta_{\alpha_2 \alpha_1} = \Theta_{\alpha_2} \cdot \Theta_{\alpha_1} - A_{\alpha_2 \alpha_1} \gamma \Theta_\gamma - \mathbf{K} \Lambda_{\alpha_2 \alpha_1} - (\Theta, \Lambda_{\alpha_2 \alpha_1})_{\hbar} \text{ mod } t_H^{n-1}.$$

Note that $\Theta_0 = 1$. Then the classical limit of quantum master equation in $\mathbf{P}(n)$ is

$$\Theta_{\alpha_2} \cdot \Theta_{\alpha_1} = A_{\alpha_2 \alpha_1} \gamma \Theta_\gamma + Q \Lambda_{\alpha_2 \alpha_1} - (\Theta, \Lambda_{\alpha_2 \alpha_1}) \text{ mod } t_H^{n-1}, \quad (5.1)$$

where $\Theta_{\alpha_1} = \Theta_{\alpha_1}|_{\hbar=0}$ and $\Lambda_{\alpha_2 \alpha_1} = \Lambda_{\alpha_2 \alpha_1}|_{\hbar=0}$. We may also use the following decompositions

$$\Theta_\alpha = \Theta_\alpha^{[0]} + \Theta_\alpha^{[1]} + \dots + \Theta_\alpha^{[n-1]},$$

where $\Theta_\alpha^{[k-1]} = \partial_\alpha \Theta^{[k]}$. Note that $\Theta_\alpha^{[0]} = \mathbf{O}_\alpha$ and $t^\alpha \Theta_\alpha^{[k-1]} = k \Theta^{[k]}$, since $\Theta^{[k]}$ is a homogeneous polynomial in degree k in t_H .

Now we turn to the quantum descendant equation in $\mathbf{P}(n)$: $\mathbf{K}\Theta + \frac{1}{2}(\Theta, \Theta)_{\hbar} = 0 \text{ mod } t_H^{n+1}$.

Corollary 5.1. *Let $\mathbf{K}_\Theta := \mathbf{K} + (\Theta, \cdot)_{\hbar} \text{ mod } t_H^{n+1}$. On any $\mathbf{M} \in \mathcal{C}[[t_H, \hbar]]][[\hbar]] \text{ mod } t_H^m$ for $m \leq n+1$,*

$$\mathbf{K}_\Theta^2 \mathbf{M} = 0 \text{ mod } t_H^m$$

Proof. Note that $\mathbf{K}_\Theta \mathbf{M} \equiv \mathbf{K} \mathbf{M} + (\Theta, \mathbf{M})_{\hbar} \text{ mod } t_H^m$, and

$$\mathbf{K}_\Theta^2 \mathbf{M} = \mathbf{K}^2 \mathbf{M} + \mathbf{K}(\Theta, \mathbf{M})_{\hbar} + (\Theta, \mathbf{K} \mathbf{M})_{\hbar} + (\Theta, (\Theta, \mathbf{M})_{\hbar})_{\hbar} \text{ mod } t_H^m$$

Using $\mathbf{K}^2 = 0$, the property that \mathbf{K} is a derivation of the BV bracket and the Jacobi identity of the BV bracket, we have

$$\mathbf{K}_\Theta^2 \mathbf{M} = \left(\mathbf{K}\Theta + \frac{1}{2}(\Theta, \Theta)_{\hbar}, \mathbf{M} \right)_{\hbar} \text{ mod } t_H^m.$$

Since $\mathbf{K}\Theta + \frac{1}{2}(\Theta, \Theta)_{\hbar} = 0 \text{ mod } t_H^{n+1}$ and $m \leq n+1$, we conclude that

$$\mathbf{K}_\Theta^2 \mathbf{M} = 0 \text{ mod } t_H^m.$$

□

By applying ∂_{α_1} and ∂_{α_2} successively to the quantum descendant equation, we also have

Corollary 5.2. $K\Theta_{\alpha_1} + (\Theta, \Theta_{\alpha_1})_{\hbar} = 0 \mod t_H^n$.

Let $\Theta = \Theta|_{\hbar=0}$ and let $\Theta_\gamma = \Theta_\gamma|_{\hbar=0}$. By taking the classical limit of the quantum descendant equation in $\mathbf{P}(n)$ we have

Corollary 5.3. $Q\Theta + \frac{1}{2}(\Theta, \Theta) = 0 \mod t_H^{n+1}$

By taking the classical limit of corollary 5.1, we have

Corollary 5.4. Let $Q_\Theta := Q + (\Theta,) \mod t_H^{n+1}$. On any $M \in \mathcal{C}[[t_H]] \mod t_H^m$ for $m \leq n+1$,

$$Q_\Theta^2 M = 0 \mod t_H^m.$$

By taking the classical limit of corollary 5.2, we have

Corollary 5.5. $Q\Theta_\gamma + (\Theta, \Theta_\gamma) = 0 \mod t_H^n$

The condition $Q\Theta_\gamma + (\Theta, \Theta_\gamma) = 0 \mod t_H^n$ implies that $Q\Theta_\gamma + (\Theta, \Theta_\gamma) = 0 \mod t_H^k$ for all $1 \leq k \leq n$. In particular $Q\Theta_\gamma + (\Theta, \Theta_\gamma) = 0 \mod t_H^1$ is equivalent to $QO_\gamma = 0$. By our assumption the Q -cohomology classes of $\{O_\gamma\}$ form a (linearly independent) basis of H , any homogeneous element $X \in \mathcal{C}[[t_H]]^{|X|}$ satisfying $QX = 0$ can be expressed as $X = c^\gamma O_\gamma + QY$ for unique set of constants $\{c^\gamma\}$ in \mathbb{k} and some $Y \in \mathcal{C}[[t_H]]^{|X|-1}$ defined modulo $\text{Ker } Q$. Also for any equality in the form $QY = c^\gamma O_\gamma$ implies that $c^\gamma = 0$ for all γ , since by taking the Q -cohomology class we have $c^\gamma [O_\gamma] \equiv c^\gamma e_\gamma = 0$ and $\{e_\gamma\}$ are linearly independent, as well as $QY = 0$. We shall establish the similar properties involving $Q + (\Theta,)$ and $\{\Theta\}_\alpha \mod t^n$ as consequences of the $\mathbf{P}(n)$.

Proposition 5.1. Any element $M \in \mathcal{C}[[t_H]]^{|M|}$ defined modulo t_H^m for $1 \leq m \leq n$ satisfying

$$QM + (\Theta, M) = 0 \mod t_H^m$$

can be expressed as

$$M = B^\gamma \Theta_\gamma + Q\Lambda + (\Theta, \Lambda)_\hbar \mod t_H^m$$

where $B^\gamma \in \mathbb{k}[[t_H]] \mod t_H^m$ and $\Lambda \in \mathcal{C}[[t_H]]^{|M|-1} \mod t_H^m$ defined modulo $\text{Ker}(Q + (\Theta,))$.

Proof. See Appendix B. \square

Corollary 5.6. For any $M \in \mathcal{C}[[t_H]]^{|M|}$ defined modulo t_H^m for $1 \leq m \leq n$ for $1 \leq m \leq n$ satisfying

$$QM + (\Theta, M) = 0 \text{ mod } t_H^m,$$

there is a canonical extension to $\mathbf{M} \in \mathcal{C}[[t_H, \hbar]]^{|M|} \text{ mod } t_H^m$, which is unique up to $\hbar \text{Im } \mathbf{K}_\Theta$, such that $\mathbf{M}|_{\hbar=0} = M$ and

$$\mathbf{K}\mathbf{M} + (\Theta, \mathbf{M})_\hbar = 0 \text{ mod } t_H^m.$$

Proof. From proposition 5.1, M can be expressed as

$$M = B^\gamma \Theta_\gamma + Q\Lambda + (\Theta, \Lambda) \text{ mod } t_H^m$$

for unique $B^\gamma \in \mathbb{K}[t_H] \text{ mod } t_H^m$ and some $\Lambda \in (\mathcal{C} \otimes \mathbb{K}[t_H])^{|M|-1} \text{ mod } t_H^m$ defined modulo $\text{Ker}(Q + (\Theta, \cdot))$. Let $\Lambda \in \mathcal{C}[[t_H, \hbar]]^{|M|-1} \text{ mod } t_H^m$ such that $\Lambda|_{\hbar=0} = \Lambda$, and let

$$\mathbf{M} = B^\gamma \Theta_\gamma + \mathbf{K}\Lambda + (\Theta, \Lambda)_\hbar \text{ mod } t_H^m,$$

Then $\mathbf{M}|_{\hbar=0} = M$ and $\mathbf{K}\mathbf{M} + (\Theta, \mathbf{M})_\hbar = 0 \text{ mod } t_H^m$, where we have used corollaries 5.1 and 5.2.

□

Proposition 5.2. Let $m \leq n$. An equality in the following form:

$$QM + (\Theta, M) = C^\gamma \Theta_\gamma \text{ mod } t_H^m,$$

where $M \in \mathcal{C}[[t_H]]^{|M|}$ and $C^\gamma \in \mathbb{K}[[t_H]] \text{ mod } t_H^m$, implies that

$$QM + (\Theta, M) = C^\gamma = 0 \text{ mod } t_H^m.$$

Proof. See Appendix B. □

Now consider the following two expressions, which are defined modulo t_H^{n-1} and built from $\mathbf{P}(n)$:

$$\begin{aligned} \mathbb{M}_{\alpha_3 \alpha_2 \alpha_1} &:= \Theta_{\alpha_3 \alpha_2} \cdot \Theta_{\alpha_1} + (-1)^{|\alpha_3||\alpha_2|} \Theta_{\alpha_2} \cdot \Theta_{\alpha_3 \alpha_1} - A_{\alpha_2 \alpha_1}^\gamma \Theta_{\alpha_3 \gamma} - (\Theta_{\alpha_3}, \Lambda_{\alpha_2 \alpha_1})_\hbar \\ &\text{mod } t_H^{n-1}, \\ \mathbb{N}_{\alpha_3 \alpha_2 \alpha_1} &:= -A_{\alpha_2 \alpha_1}^\rho \Lambda_{\alpha_3 \rho} + (-1)^{|\alpha_3||\alpha_2|} A_{\alpha_3 \alpha_1}^\rho \Lambda_{\alpha_2 \rho} \\ &\quad - (-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2 \alpha_1} + (-1)^{|\alpha_3||\alpha_2|} (-1)^{|\alpha_2|} \Theta_{\alpha_2} \cdot \Lambda_{\alpha_3 \alpha_1} \\ &\text{mod } t_H^{n-1}. \end{aligned} \tag{5.2}$$

Then,

Proposition 5.3. *The two expressions $\mathbb{M}_{\alpha_3\alpha_2\alpha_1}$ and $\mathbb{N}_{\alpha_3\alpha_2\alpha_1}$ satisfy*

1. $\mathbb{M}_{\alpha_3\alpha_2\alpha_1} = (-1)^{|\alpha_2||\alpha_1|} \mathbb{M}_{\alpha_3\alpha_1\alpha_2} \mod t_H^{n-1},$
2. $\mathbb{M}_{\alpha_3\alpha_20} = \mathbb{M}_{\alpha_30\alpha_1} = \mathbb{M}_{0\alpha_2\alpha_1} = 0 \mod t_H^{n-1},$
3. $\mathbb{N}_{\alpha_3\alpha_2\alpha_1} = -(-1)^{|\alpha_3||\alpha_2|} \mathbb{N}_{\alpha_2\alpha_3\alpha_1} \mod t_H^{n-1}$
4. $\mathbb{N}_{\alpha_3\alpha_20} = \mathbb{N}_{\alpha_30\alpha_1} = \mathbb{N}_{0\alpha_2\alpha_1} = 0 \mod t_H^{n-1},$
5. $\mathbb{N}_{\alpha_3\alpha_2\alpha_1} - (-1)^{|\alpha_2||\alpha_1|} \mathbb{N}_{\alpha_3\alpha_1\alpha_2} + (-1)^{|\alpha_3|(|\alpha_2|+|\alpha_1|)} \mathbb{N}_{\alpha_2\alpha_1\alpha_3} = 0 \mod t_H^{n-1},$
6. $\mathbf{K} \mathbb{N}_{\alpha_3\alpha_2\alpha_1} + (\Theta, \mathbb{N}_{\alpha_3\alpha_2\alpha_1})_{\hbar} = \hbar \left(\mathbb{M}_{\alpha_3\alpha_2\alpha_1} - (-1)^{|\alpha_3||\alpha_2|} \mathbb{M}_{\alpha_2\alpha_3\alpha_1} \right) \equiv \hbar \mathbb{M}_{[\alpha_3\alpha_2]\alpha_1} \mod t_H^{n-1}$

Proof. In order.

1. It follows from the supercommutativity of the product \cdot and the assumptions that $\Theta_{\alpha_3\alpha_1} = (-1)^{|\alpha_1||\alpha_3|} \Theta_{\alpha_1\alpha_3} \mod t_H^{n-1}$, $A_{\alpha_2\alpha_1}^\gamma = (-1)^{|\alpha_2||\alpha_1|} A_{\alpha_1\alpha_2} \mod t_H^{n-1}$ and $\Lambda_{\alpha_2\alpha_1} = (-1)^{|\alpha_2||\alpha_1|} \Lambda_{\alpha_1\alpha_2} \mod t_H^{n-1}$ in $\mathbf{P}(n)$.
2. It follows from the definition of $\mathbf{M}_{\alpha_3\alpha_2\alpha_1}$ in (5.2) together with the assumptions that $\Theta_{\beta 0} = \Theta_{0\beta} = 0 \mod t_H^{n-1}$, $\Theta_0 = 1 \mod t_H^{n-1}$, $A_{\beta 0}^\gamma = A_{0\beta}^\gamma = \delta_{\beta}^\gamma \mod t_H^{n-1}$ and $\Lambda_{\beta 0} = 0 \mod t_H^{n-1}$ in $\mathbf{P}(n)$.
3. It is trivial by definition.
4. This can be checked using the assumptions that $\Theta_0 = 1 \mod t_H^{n-1}$, $A_{\beta 0}^\gamma = A_{0\beta}^\gamma = \delta_{\beta}^\gamma \mod t_H^{n-1}$, $\Lambda_{\beta\alpha} = (-1)^{|\beta||\alpha|} \Lambda_{\alpha\beta} \mod t_H^{n-1}$ and $\Lambda_{\beta 0} = 0 \mod t_H^{n-1}$ in $\mathbf{P}(n)$.
5. This follows directly after comparison using $A_{\alpha_2\alpha_1}^\gamma = (-1)^{|\alpha_2||\alpha_1|} A_{\alpha_1\alpha_2} \mod t_H^{n-1}$ and $\Lambda_{\alpha_2\alpha_1} = (-1)^{|\alpha_2||\alpha_1|} \Lambda_{\alpha_1\alpha_2} \mod t_H^{n-1}$ in $\mathbf{P}(n)$.
6. For the last property, multiply Θ_{α_3} to the $\mod t_H^{n-1}$ quantum master equation in $\mathbf{P}(n)$ to obtain that

$$\begin{aligned} \hbar \Theta_{\alpha_3} \cdot \Theta_{\alpha_2\alpha_1} &= \Theta_{\alpha_3} \cdot (\Theta_{\alpha_2} \cdot \Theta_{\alpha_1}) - A_{\alpha_2\alpha_1}^\gamma \Theta_{\alpha_3} \cdot \Theta_\gamma - \hbar (\Theta_{\alpha_3}, \Lambda_{\alpha_2\alpha_1})_{\hbar} \\ &\quad - \mathbf{K} \left((-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2\alpha_1} \right) - (\Theta, (-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2\alpha_1})_{\hbar} \\ &\quad \mod t_H^{n-1}. \end{aligned}$$

Using the $\text{mod } t_H^{n-1}$ quantum master equation again, we have

$$\begin{aligned} \hbar \Theta_{\alpha_3} \cdot \Theta_{\alpha_2 \alpha_1} &= \Theta_{\alpha_3} \cdot (\Theta_{\alpha_2} \cdot \Theta_{\alpha_1}) - A_{\alpha_2 \alpha_1}{}^\rho A_{\alpha_3 \rho}{}^\gamma \Theta_\gamma - \hbar A_{\alpha_2 \alpha_1}{}^\rho \Theta_{\alpha_3 \rho} - \hbar (\Theta_{\alpha_3}, \Lambda_{\alpha_2 \alpha_1})_\hbar \\ &\quad - \mathbf{K} \left(A_{\alpha_2 \alpha_1}{}^\rho \Lambda_{\alpha_3 \rho} + (-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2 \alpha_1} \right) \\ &\quad - \left(\Theta, A_{\alpha_2 \alpha_1}{}^\rho \Lambda_{\alpha_3 \rho} + (-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2 \alpha_1} \right)_\hbar \\ &\quad \text{mod } t_H^{n-1}. \end{aligned}$$

Equivalently

$$\begin{aligned} \Theta_{\alpha_3} \cdot (\Theta_{\alpha_2} \cdot \Theta_{\alpha_1}) &= A_{\alpha_2 \alpha_1}{}^\rho A_{\alpha_3 \rho}{}^\gamma \Theta_\gamma + \hbar \Theta_{\alpha_3} \cdot \Theta_{\alpha_2 \alpha_1} + \hbar A_{\alpha_2 \alpha_1}{}^\rho \Theta_{\alpha_3 \rho} + \hbar (\Theta_{\alpha_3}, \Lambda_{\alpha_2 \alpha_1}) \\ &\quad - \mathbf{K} \left(A_{\alpha_2 \alpha_1}{}^\rho \Lambda_{\alpha_3 \rho} - (-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2 \alpha_1} \right) \\ &\quad - \left(\Theta, A_{\alpha_2 \alpha_1}{}^\rho \Lambda_{\alpha_3 \rho} - (-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2 \alpha_1} \right)_\hbar \\ &\quad \text{mod } t_H^{n-1}. \end{aligned}$$

Then from the identity

$$\Theta_{\alpha_3} \cdot (\Theta_{\alpha_2} \cdot \Theta_{\alpha_1}) - (-1)^{|\alpha_3||\alpha_2|} \Theta_{\alpha_2} \cdot (\Theta_{\alpha_3} \cdot \Theta_{\alpha_1}) = 0 \text{ mod } t_H^{n-1},$$

we have

$$\begin{aligned} 0 &= \left(A_{\alpha_2 \alpha_1}{}^\rho A_{\alpha_3 \rho}{}^\gamma - (-1)^{|\alpha_3||\alpha_2|} A_{\alpha_3 \alpha_1}{}^\rho A_{\alpha_2 \rho}{}^\gamma \right) \Theta_\gamma \\ &\quad + \hbar \left(\Theta_{\alpha_3} \cdot \Theta_{\alpha_2 \alpha_1} - (-1)^{|\alpha_3||\alpha_2|} \Theta_{\alpha_2} \cdot \Theta_{\alpha_3 \alpha_1} \right) \\ &\quad + \hbar \left(A_{\alpha_2 \alpha_1}{}^\gamma \Theta_{\alpha_3 \gamma} - (-1)^{|\alpha_3||\alpha_2|} A_{\alpha_3 \alpha_1}{}^\gamma \Theta_{\alpha_2 \gamma} \right) \\ &\quad + \hbar \left((\Theta_{\alpha_3}, \Lambda_{\alpha_2 \alpha_1})_\hbar - (-1)^{|\alpha_3||\alpha_2|} (\Theta_{\alpha_2}, \Lambda_{\alpha_3 \alpha_1})_\hbar \right) \\ &\quad - \mathbf{K} \left(-A_{\alpha_2 \alpha_1}{}^\rho \Lambda_{\alpha_3 \rho} + (-1)^{|\alpha_3||\alpha_2|} A_{\alpha_3 \alpha_1}{}^\rho \Lambda_{\alpha_2 \rho} - (-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2 \alpha_1} + (-1)^{|\alpha_3||\alpha_2|} (-1)^{|\alpha_2|} \Theta_{\alpha_2} \cdot \Lambda_{\alpha_3 \alpha_1} \right) \\ &\quad - \left(\Theta, -A_{\alpha_2 \alpha_1}{}^\rho \Lambda_{\alpha_3 \rho} + (-1)^{|\alpha_3||\alpha_2|} A_{\alpha_3 \alpha_1}{}^\rho \Lambda_{\alpha_2 \rho} - (-1)^{|\alpha_3|} \Theta_{\alpha_3} \cdot \Lambda_{\alpha_2 \alpha_1} + (-1)^{|\alpha_3||\alpha_2|} (-1)^{|\alpha_2|} \Theta_{\alpha_2} \cdot \Lambda_{\alpha_3 \alpha_1} \right)_\hbar \\ &\quad \text{mod } t_H^{n-1}. \end{aligned}$$

Now we use that $A_{\alpha_2 \alpha_1}{}^\rho A_{\alpha_3 \rho}{}^\gamma - (-1)^{|\alpha_3||\alpha_2|} A_{\alpha_3 \alpha_1}{}^\rho A_{\alpha_2 \rho}{}^\gamma = 0 \text{ mod } t_H^{n-1}$ and compare the remaining terms with the definitions of $\mathbb{N}_{\alpha_3 \alpha_2 \alpha_1} \text{ mod } t_H^{n-1}$ and $\mathbb{M}_{\alpha_3 \alpha_2 \alpha_1} \text{ mod } t_H^{n-1}$ in the equation (5.2) to establish the property. \square

Finally, we consider the quantum gauge the 3rd condition for the quantum gauge (property 8 of $\mathbf{P}(n)$), which can be written as follows:

$$\hbar \partial_{[\tilde{\alpha}_3} \Lambda_{\alpha_2] \alpha_1} = \mathbb{N}_{\alpha_3 \alpha_2 \alpha_1} - B_{\alpha_3 \alpha_2 \alpha_1}{}^\rho \Theta_\rho - \mathbf{K} X_{\alpha_3 \alpha_2 \alpha_1} - (\Theta, X_{\alpha_3 \alpha_2 \alpha_1})_\hbar \text{ mod } t_H^{n-2}, \quad (5.3)$$

where $\mathbb{N}_{\alpha_3 \alpha_2 \alpha_1}$ contributes to the above modulo t_H^{n-2} only.

5.2. Building $\mathbf{P}(n+1)$ out of $\mathbf{P}(n)$

Our goal in this subsection is to build $\mathbf{P}(n+1)$ out of assumed $\mathbf{P}(n)$. We are going to proceed in the following orders: (1) define $\tilde{A}_{\alpha\beta}^\gamma$ such that $\tilde{A}_{0\beta}^\gamma = \delta_\beta^\gamma$ and $\tilde{A}_{\alpha\beta}^\gamma = A_{\alpha\beta}^\gamma \bmod t_H^{n-1}$, (2) establish the $\bmod t_H^n$ graded commutativity and associativity of $\tilde{A}_{\alpha\beta}^\gamma$, (3) establish the $\bmod t_H^{n-1}$ potentiality of $\tilde{A}_{\alpha\beta}^\gamma$, (4) define $\tilde{\Theta}$ such that $\partial_0 \tilde{\Theta} = 1$ and $\tilde{\Theta} = \Theta \bmod t_H^{n+1}$ and show that $\tilde{\Theta}$ satisfy the $\bmod t_H^n$ quantum master equation and the $\bmod t_H^{n+2}$ quantum descendant equation.

5.2.1. Definition of $\tilde{A}_{\alpha\beta}^\gamma$ and its immediate properties. We first consider the following expression, built from $\mathbf{P}(n)$,

$$\mathbb{M}_{\alpha_2\alpha_1} := \Theta_{\alpha_2} \cdot \Theta_{\alpha_1} \bmod t_H^n, \quad (5.4)$$

which is defined modulo t_H^n since Θ_α are defined modulo t_H^n in $\mathbf{P}(n)$. Then, we obtain that

$$\mathbf{KM}_{\alpha_2\alpha_1} + (\Theta, \mathbb{M}_{\alpha_2\alpha_1}) = -\hbar(-1)^{|\alpha_2|}(\Theta_{\alpha_2}, \Theta_{\alpha_1})_{\hbar} \bmod t_H^n \quad (5.5)$$

by a direct computation.

Lemma 5.1. *There are unique $\tilde{A}_{\alpha_2\alpha_1}^\gamma \in \mathbb{k}[[t_H]] \bmod t_H^n$ and some $\tilde{\Lambda}_{\alpha_2\alpha_1} \in \mathcal{C}[[t_H]]^{|\alpha_2|+|\alpha_1|-1} \bmod t_H^n$ satisfying*

$$\Theta_{\alpha_2} \cdot \Theta_{\alpha_1} = \tilde{A}_{\alpha_2\alpha_1}^\gamma \Theta_\gamma + Q\tilde{\Lambda}_{\alpha_2\alpha_1} + (\Theta, \tilde{\Lambda}_{\alpha_2\alpha_1}) \bmod t_H^n, \quad (5.6)$$

with the following properties

1. $\tilde{A}_{\alpha_2\alpha_1}^\gamma - (-1)^{|\alpha_2||\alpha_1|} \tilde{A}_{\alpha_1\alpha_2}^\gamma = 0 \bmod t_H^n$,
2. $\tilde{A}_{0\alpha_1}^\gamma = \delta_{\alpha_1}^\gamma \bmod t_H^n$,
3. $\tilde{A}_{\alpha_2\alpha_1}^\rho \tilde{A}_{\alpha_3\rho}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \tilde{A}_{\alpha_3\alpha_1}^\rho \tilde{A}_{\alpha_2\rho}^\gamma = 0 \bmod t_H^n$,
4. $\tilde{\Lambda}_{\alpha_2\alpha_1} = (-1)^{|\alpha_2||\alpha_1|} \tilde{\Lambda}_{\alpha_1\alpha_2} \bmod t_H^n$,
5. $\tilde{\Lambda}_{0\alpha_1} = 0 \bmod t_H^n$.

Proof. By taking classical limit of (5.2), we obtain that $Q\mathcal{M}_{\alpha_2\alpha_1} + (\Theta, \mathcal{M}_{\alpha_2\alpha_1}) = 0 \bmod t_H^n$, where $\mathcal{M}_{\alpha_2\alpha_1} := \mathbb{M}_{\alpha_2\alpha_1}|_{t_H=0} = \Theta_{\alpha_2} \cdot \Theta_{\alpha_1} \in \mathcal{C}[[t_H]]^{|\alpha_2|+|\alpha_1|} \bmod t_H^n$. Applying proposition 5.1, we conclude that there are unique 3-tensor $\tilde{A}_{\alpha_2\alpha_1}^\gamma \in \mathbb{k}[[t_H]] \bmod t_H^n$ and some $\tilde{X}_{\alpha_2\alpha_1} \in \mathcal{C}[[t_H]]^{|\alpha_2|+|\alpha_1|} \bmod t_H^n$, defined modulo $\text{Ker } Q_\Theta$, such that

$$\Theta_{\alpha_2} \cdot \Theta_{\alpha_1} = \tilde{A}_{\alpha_2\alpha_1}^\gamma \Theta_\gamma + Q\tilde{X}_{\alpha_2\alpha_1} + (\Theta, \tilde{X}_{\alpha_2\alpha_1}) \bmod t_H^n. \quad (5.7)$$

From the super-commutativity $\Theta_{a_2} \cdot \Theta_{a_1} - (-1)^{|\alpha_2||\alpha_1|} \Theta_{a_1} \cdot \Theta_{a_2} = 0 \mod t_H^n$, the equation (5.7) implies that

$$\left(\tilde{A}_{a_2 a_1}^\gamma - (-1)^{|\alpha_2||\alpha_1|} \tilde{A}_{a_1 a_2}^\gamma \right) \Theta_\gamma = -Q \tilde{X}_{[a_2 a_1]} - \left(\Theta, \tilde{X}_{[a_2 a_1]} \right) \mod t_H^n.$$

where $\tilde{X}_{[a_2 a_1]} := \tilde{X}_{a_2 a_1} - (-1)^{|\alpha_2||\alpha_1|} \tilde{X}_{a_1 a_2} \mod t_H^n$. We now apply proposition 5.2 to conclude that

$$\tilde{A}_{a_2 a_1}^\gamma - (-1)^{|\alpha_2||\alpha_1|} \tilde{A}_{a_1 a_2}^\gamma = 0 \mod t_H^n$$

and $Q \tilde{X}_{[a_2 a_1]} + \left(\Theta, \tilde{X}_{[a_2 a_1]} \right) = 0$. Let

$$\tilde{\Lambda}_{a_2 a_1} := \frac{1}{2} \left(\tilde{X}_{a_2 a_1} + (-1)^{|\alpha_2||\alpha_1|} \tilde{X}_{a_1 a_2} \right) \mod t_H^n,$$

such that $\tilde{\Lambda}_{a_2 a_1} = (-1)^{|\alpha_2||\alpha_1|} \tilde{\Lambda}_{a_1 a_2} \mod t_H^n$. Then (5.7) is rewritten as follows

$$\Theta_{a_2} \cdot \Theta_{a_1} = \tilde{A}_{a_2 a_1}^\gamma \Theta_\gamma + Q \tilde{\Lambda}_{a_2 a_1} + \left(\Theta, \tilde{\Lambda}_{a_2 a_1} \right) \mod t_H^n.$$

Hence we have established the relation (5.6) as well as the properties 1 and 4. For the properties 2 and 5, set $a_2 = 0$ in the above equation and use $\Theta_0 = 1$ to deduce that $\tilde{A}_{0\beta}^\gamma = \delta_\beta^\gamma$ and $\tilde{\Lambda}_{0a_1} = 0 \mod t_H^n$.

For the last remaining property 3, consider the following identity

$$\Theta_{a_3} \cdot (\Theta_{a_2} \cdot \Theta_{a_1}) - (-1)^{|\alpha_3||\alpha_2|} \Theta_{a_2} \cdot (\Theta_{a_3} \cdot \Theta_{a_1}) = 0, \mod t_H^n \quad (5.8)$$

which is due to the associativity and the super-commutativity of the product. Using the relation (5.6), we have

$$\Theta_{a_3} \cdot (\Theta_{a_2} \cdot \Theta_{a_1}) = \tilde{A}_{a_2 a_1}^\rho \Theta_{a_3} \cdot \Theta_\rho + Q_\Theta \left((-1)^{|\alpha_3|} \Theta_{a_3} \cdot \tilde{\Lambda}_{a_2 a_1} \right) \mod t_H^n,$$

where we used the property that $Q_\Theta \mod t_H^{n+1}$ is a derivation of the product and $Q_\Theta \Theta_a = 0 \mod t_H^n$. Using the relation (5.6) again for $\Theta_{a_3} \cdot \Theta_\rho$, we have

$$\Theta_{a_3} \cdot (\Theta_{a_2} \cdot \Theta_{a_1}) = \tilde{A}_{a_2 a_1}^\rho \tilde{A}_{a_3 \rho}^\gamma \Theta_\gamma + Q_\Theta \left(\tilde{A}_{a_2 a_1}^\rho \tilde{\Lambda}_{\rho a_3} + (-1)^{|\alpha_3|} \Theta_{a_3} \cdot \tilde{\Lambda}_{a_2 a_1} \right) \mod t_H^n.$$

Then the identity (5.8) imply that

$$\left(\tilde{A}_{a_2 a_1}^\rho \tilde{A}_{a_3 \rho}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \tilde{A}_{a_3 a_1}^\rho \tilde{A}_{a_2 \rho}^\gamma \right) \Theta_\gamma = -Q \tilde{\mathcal{N}}_{a_3 a_2 a_1} - \left(\Theta, \tilde{\mathcal{N}}_{a_3 a_2 a_1} \right) \mod t_H^n, \quad (5.9)$$

where

$$\begin{aligned} \tilde{\mathcal{N}}_{a_3 a_2 a_1} &:= \tilde{A}_{a_2 a_1}^\rho \tilde{\Lambda}_{a_3 \rho} - (-1)^{|\alpha_3||\alpha_2|} \tilde{A}_{a_3 a_1}^\rho \tilde{X}_{a_2 \rho} \\ &\quad - (-1)^{|\alpha_3|} \Theta_{a_3} \cdot \tilde{\Lambda}_{a_2 a_1} + (-1)^{|\alpha_3||\alpha_2|} (-1)^{|\alpha_2|} \Theta_{a_2} \cdot \tilde{\Lambda}_{a_3 a_1} \mod t_H^n. \end{aligned}$$

Applying proposition 5.2 in Appendix A to the equation (5.9) to conclude that

$$\tilde{A}_{a_2 a_1}^\rho \tilde{A}_{a_3 \rho}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \tilde{A}_{a_3 a_1}^\rho \tilde{A}_{a_2 \rho}^\gamma = 0 \mod t_H^n.$$

□

Corollary 5.7. *Associativity modulo t_H^n*

$$\tilde{A}_{\alpha_3\alpha_2}{}^\rho \tilde{A}_{\rho\alpha_1}{}^\gamma - \tilde{A}_{\alpha_2\alpha_1}{}^\rho \tilde{A}_{\alpha_3\rho}{}^\gamma = 0 \mod t_H^n.$$

Proof. Combine properties 1 and 3 in lemma 5.1. \square

Corollary 5.8. *Reduction modulo t_H^{n-1} : $\tilde{A}_{\alpha_2\alpha_1}{}^\gamma = A_{\alpha_2\alpha_1}{}^\gamma \mod t_H^{n-1}$ and $\tilde{\Lambda}_{\alpha_2\alpha_1} = \Lambda_{\alpha_2\alpha_1} \mod t_H^{n-1}$.*

Proof. Compare (5.6) with the classical limit (5.1) of quantum master equation in $\mathbf{P}(n)$.

5.2.2. *Potentiality $\tilde{A}_{\alpha\beta}{}^\gamma$.* Here is the idea of our proof. Recall the defining equation (5.6) of $\tilde{A}_{\alpha\beta}{}^\gamma$ in lemma 5.1:

$$\Theta_{\alpha_2} \cdot \Theta_{\alpha_1} = \tilde{A}_{\alpha_2\alpha_1}{}^\gamma \Theta_\gamma + Q \tilde{\Lambda}_{\alpha_2\alpha_1} + (\Theta, \tilde{\Lambda}_{\alpha_2\alpha_1}) \mod t_H^n,$$

where $\tilde{A}_{\alpha_2\alpha_1}{}^\gamma = A_{\alpha_2\alpha_1}{}^\gamma \mod t_H^{n-1}$ and $\tilde{\Lambda}_{\alpha_2\alpha_1} = \Lambda_{\alpha_2\alpha_1} \mod t_H^{n-1}$. Apply ∂_{α_3} to the above equation to have

$$\begin{aligned} \Theta_{\alpha_3\alpha_2} \cdot \Theta_{\alpha_1} + (-1)^{|\alpha_3||\alpha_2|} \Theta_{\alpha_2} \cdot \Theta_{\alpha_3\alpha_1} - \tilde{A}_{\alpha_2\alpha_1}{}^\gamma \Theta_{\alpha_3\gamma} - (\Theta_{\alpha_3}, \tilde{\Lambda}_{\alpha_2\alpha_1}) \\ = \partial_{\alpha_3} \tilde{A}_{\alpha_2\alpha_1}{}^\gamma \Theta_\gamma + Q_\Theta (-1)^{|\alpha_3|} \partial_{\alpha_3} \tilde{\Lambda}_{\alpha_2\alpha_1} \mod t_H^{n-1}. \end{aligned}$$

Denote the LHS of above by $\mathcal{M}_{\alpha_3\alpha_2\alpha_1}$:

$$\mathcal{M}_{\alpha_3\alpha_2\alpha_1} = \Theta_{\alpha_3\alpha_2} \cdot \Theta_{\alpha_1} + (-1)^{|\alpha_3||\alpha_2|} \Theta_{\alpha_2} \cdot \Theta_{\alpha_3\alpha_1} - A_{\alpha_2\alpha_1}{}^\gamma \Theta_{\alpha_3\gamma} - (\Theta_{\alpha_3}, \Lambda_{\alpha_2\alpha_1}) \mod t_H^{n-1},$$

since $\tilde{A}_{\alpha_2\alpha_1}{}^\gamma = A_{\alpha_2\alpha_1}{}^\gamma \mod t_H^{n-1}$ and $\tilde{\Lambda}_{\alpha_2\alpha_1} = \Lambda_{\alpha_2\alpha_1} \mod t_H^{n-1}$. Hence we have

$$\begin{aligned} \mathcal{M}_{[\alpha_3\alpha_2]\alpha_1} &= (\partial_{\alpha_3} \tilde{A}_{\alpha_2\alpha_1}{}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \partial_{\alpha_2} \tilde{A}_{\alpha_3\alpha_1}{}^\gamma) \Theta_\gamma \\ &\quad + Q_\Theta \left((-1)^{|\alpha_3|} \partial_{\alpha_3} \tilde{\Lambda}_{\alpha_2\alpha_1} - (-1)^{|\alpha_2|+|\alpha_3||\alpha_2|} \partial_{\alpha_3} \tilde{\Lambda}_{\alpha_2\alpha_1} \right) \mod t_H^{n-1}. \end{aligned} \quad (5.10)$$

Thus we need to show that $\mathcal{M}_{[\alpha_3\alpha_2]\alpha_1} \in \text{Im } Q_\Theta$ to establish that

$$\partial_{\alpha_3} \tilde{A}_{\alpha_2\alpha_1}{}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \partial_{\alpha_2} \tilde{A}_{\alpha_3\alpha_1}{}^\gamma = 0 \mod t_H^{n-1}.$$

Note that $\mathcal{M}_{\alpha_3\alpha_2\alpha_1}$ is exactly the classical limit of $\mathbb{M}_{\alpha_3\alpha_2\alpha_1}$, the properties of which are listed in proposition 5.3. We shall need a stronger proposition.

Proposition 5.4. *There is certain $\tilde{\mathbb{Y}}_{a_3 a_2 a_1} \in \mathcal{C}[[t_H, \hbar]]^{|\alpha_3|+|\alpha_2|+|\alpha_1|-1} \bmod t_H^{n-1}$ such that*

$$\tilde{\mathbb{Y}}_{a_3 a_2 a_1} = \partial_{[\tilde{a}_3 \Lambda_{a_2}] a_1} \bmod t_H^{n-2},$$

and

1. $\mathbf{K} \tilde{\mathbb{Y}}_{a_3 a_2 a_1} + (\Theta, \tilde{\mathbb{Y}}_{a_3 a_2 a_1})_{\hbar} = \mathbb{M}_{[a_3 a_2] a_1} \bmod t_H^{n-1},$
2. $\tilde{\mathbb{Y}}_{a_3 a_2 a_1} = -(-1)^{|\alpha_3||\alpha_2|} \tilde{\mathbb{Y}}_{a_2 a_3 a_1} \bmod t_H^{n-1},$
3. $\tilde{\mathbb{Y}}_{0 a_2 a_1} = \tilde{\mathbb{Y}}_{a_3 0 a_1} = \tilde{\mathbb{Y}}_{a_3 a_2 0} = 0 \bmod t_H^{n-1},$
4. $\tilde{\mathbb{Y}}_{a_3 a_2 a_1} - (-1)^{|\alpha_2||\alpha_1|} \tilde{\mathbb{Y}}_{a_3 a_1 a_2} + (-1)^{|\alpha_3|(|\alpha_2|+|\alpha_1|)} \tilde{\mathbb{Y}}_{a_2 a_1 a_3} = 0 \bmod t_H^{n-1}.$

Proof. Let $\mathcal{N}_{a_3 a_2 a_1} \bmod t_H^{n-1}$ be the classical limit of $\mathbb{N}_{a_3 a_2 a_1} \bmod t_H^{n-1}$. Then proposition 5.3 in the classical limit $\hbar = 0$ becomes

$$Q \mathcal{N}_{a_3 a_2 a_1} + (\Theta, \mathcal{N}_{a_3 a_2 a_1}) = 0 \bmod t_H^{n-1}.$$

Note also that

$$\begin{aligned} \mathcal{N}_{a_3 a_2 a_1} &= -(-1)^{|\alpha_3||\alpha_2|} \mathcal{N}_{a_2 a_3 a_1} \bmod t_H^{n-1}, \\ \mathcal{N}_{a_3 a_2 0} &= \mathcal{N}_{a_3 0 a_1} = \mathcal{N}_{0 a_2 a_1} = 0 \bmod t_H^{n-1}, \\ \mathcal{N}_{a_3 a_2 a_1} - (-1)^{|\alpha_2||\alpha_1|} \mathcal{N}_{a_3 a_1 a_2} + (-1)^{|\alpha_3|(|\alpha_2|+|\alpha_1|)} \mathcal{N}_{a_2 a_1 a_3} &= 0 \bmod t_H^{n-1}, \end{aligned}$$

which are the classical limits of properties 3, 4 and 5 in proposition 5.3. We also recall the 3rd condition for the quantum gauge (property 8 of $\mathbf{P}(n)$ as is written in (5.3)):

$$\hbar \partial_{[\tilde{a}_3 \Lambda_{a_2}] a_1} = \mathbb{N}_{a_3 a_2 a_1} - B_{a_3 a_2 a_1} \gamma \Theta_{\gamma} - \mathbf{K} X_{a_3 a_2 a_1} - (\Theta, X_{a_3 a_2 a_1})_{\hbar} \bmod t_H^{n-2}, \quad (5.11)$$

which classical limit is

$$\mathcal{N}_{a_3 a_2 a_1} = B_{a_3 a_2 a_1} \gamma \Theta_{\gamma} - Q X_{a_3 a_2 a_1} - (\Theta, X_{a_3 a_2 a_1}) \bmod t_H^{n-2}.$$

Hence, by proposition 5.1, there are unique $\tilde{B}_{a_3 a_2 a_1} \in \mathbb{K}[[t_H]] \bmod t_H^{n-1}$ and some $\tilde{X}_{a_3 a_2 a_1} \in \mathcal{C}[[t_H]] \bmod t_H^{n-1}$ defined modulo $\text{Ker } Q_{\Theta}$ satisfying

$$\mathcal{N}_{a_3 a_2 a_1} = \tilde{B}_{a_3 a_2 a_1} \gamma \Theta_{\gamma} + Q \tilde{X}_{a_3 a_2 a_1} + (\Theta, \tilde{X}_{a_3 a_2 a_1}) \bmod t_H^{n-1}, \quad (5.12)$$

such that $\tilde{B}_{a_3 a_2 a_1} \gamma = B_{a_3 a_2 a_1} \bmod t_H^{n-2}$, $\tilde{\Lambda}_{a_3 a_2 a_1} \gamma = \Lambda_{a_3 a_2 a_1} \bmod t_H^{n-2}$ and

$$\begin{aligned} \tilde{B}_{a_3 a_2 a_1} \gamma &= -(-1)^{|\alpha_3||\alpha_2|} \tilde{B}_{a_2 a_3 a_1} \gamma \bmod t_H^{n-1}, \\ \tilde{B}_{0 a_2 a_1} \gamma &= \tilde{B}_{a_3 0 a_1} \gamma = \tilde{B}_{a_3 a_2 0} \gamma = 0 \bmod t_H^{n-1}, \\ \tilde{B}_{a_3 a_2 a_1} - (-1)^{|\alpha_2||\alpha_1|} \tilde{B}_{a_3 a_1 a_2} + (-1)^{|\alpha_3|(|\alpha_2|+|\alpha_1|)} \tilde{B}_{a_2 a_1 a_3} &= 0 \bmod t_H^{n-1}, \\ \tilde{X}_{a_3 a_2 a_1} &= -(-1)^{|\alpha_3||\alpha_2|} \tilde{X}_{a_2 a_3 a_1} \bmod t_H^{n-1}, \\ \tilde{X}_{0 a_2 a_1} &= \tilde{X}_{a_3 0 a_1} = \tilde{X}_{a_3 a_2 0} = 0 \bmod t_H^{n-1}, \\ \tilde{X}_{a_3 a_2 a_1} - (-1)^{|\alpha_2||\alpha_1|} \tilde{X}_{a_3 a_1 a_2} + (-1)^{|\alpha_3|(|\alpha_2|+|\alpha_1|)} \tilde{X}_{a_2 a_1 a_3} &= 0 \bmod t_H^{n-1}. \end{aligned} \quad (5.13)$$

It follows, from (5.12), that the following expression

$$\mathbb{N}_{a_3 a_2 a_1} - \tilde{B}_{a_3 a_2 a_1} \gamma \Theta_\gamma - \mathbf{K} \tilde{X}_{a_3 a_2 a_1} - (\Theta, \tilde{X}_{a_3 a_2 a_1})_{\hbar} \mod t_H^{n-1},$$

is divisible by \hbar . Hence we can define $\tilde{\mathbb{Y}}_{a_3 a_2 a_1}$ by the following formula:

$$\hbar \tilde{\mathbb{Y}}_{a_3 a_2 a_1} := \mathbb{N}_{a_3 a_2 a_1} - \tilde{B}_{a_3 a_2 a_1} \gamma \Theta_\gamma - \mathbf{K} \tilde{X}_{a_3 a_2 a_1} - (\Theta, \tilde{X}_{a_3 a_2 a_1})_{\hbar} \mod t_H^{n-1}, \quad (5.14)$$

such that $\tilde{\mathbb{Y}}_{a_3 a_2 a_1} = \partial_{[\tilde{a}_3 \Lambda_{a_2}] a_1} \mod t_H^{n-2}$.

Now we are ready to check all the properties in order.

1. Apply \mathbf{K}_Θ to (5.14) to obtain that

$$\hbar \mathbf{K} \tilde{\mathbb{Y}}_{a_3 a_2 a_1} + \hbar (\Theta, \tilde{\mathbb{Y}}_{a_3 a_2 a_1})_{\hbar} = \mathbf{K} \mathbb{N}_{a_3 a_2 a_1} + (\Theta, \mathbb{N}_{a_3 a_2 a_1})_{\hbar} \mod t_H^{n-1}.$$

Then, from property 6 in proposition 5.3, we conclude that

$$\mathbf{K} \tilde{\mathbb{Y}}_{a_3 a_2 a_1} + (\Theta, \tilde{\mathbb{Y}}_{a_3 a_2 a_1})_{\hbar} = \mathbb{M}_{[\alpha_3 a_2] a_1} \mod t_H^{n-1}.$$

2. From property 3 in proposition 5.3 and the relations in (5.13), we conclude that

$$\tilde{\mathbb{Y}}_{a_3 a_2 a_1} = -(-1)^{|\alpha_3| |\alpha_2|} \tilde{\mathbb{Y}}_{a_2 a_3 a_1} \mod t_H^{n-1}.$$

3. From property 4 in proposition 5.3 and the relations in (5.13), we conclude that

$$\tilde{\mathbb{Y}}_{0 a_2 a_1} = \tilde{\mathbb{Y}}_{a_3 0 a_1} = \tilde{\mathbb{Y}}_{a_3 a_2 0} = 0 \mod t_H^{n-1}.$$

4. From property 5 in proposition 5.3 and the relations in (5.13), we conclude that

$$\tilde{\mathbb{Y}}_{a_3 a_2 a_1} - (-1)^{|\alpha_2| |\alpha_1|} \tilde{\mathbb{Y}}_{a_3 a_1 a_2} + (-1)^{|\alpha_3| (|\alpha_2| + |\alpha_1|)} \tilde{\mathbb{Y}}_{a_2 a_1 a_3} = 0 \mod t_H^{n-1}.$$

□

Now we are ready to establish the potentiality of $\tilde{A}_{a_2 a_1} \gamma$.

Lemma 5.2. *Potentiality:* $\partial_{a_3} \tilde{A}_{a_2 a_1} \gamma - (-1)^{|\alpha_3| |\alpha_2|} \partial_{a_2} \tilde{A}_{a_3 a_1} \gamma = 0 \mod t_H^{n-1}$.

Proof. Recall the the relation (5.10):

$$\begin{aligned} \mathcal{M}_{[\alpha_3 a_2] a_1} &= (\partial_{a_3} \tilde{A}_{a_2 a_1} \gamma - (-1)^{|\alpha_3| |\alpha_2|} \partial_{a_2} \tilde{A}_{a_3 a_1} \gamma) \Theta_\gamma \\ &\quad + Q_\Theta \left((-1)^{|\alpha_3|} \partial_{a_3} \tilde{\Lambda}_{a_2 a_1} - (-1)^{|\alpha_2| + |\alpha_3| |\alpha_2|} \partial_{a_3} \tilde{\Lambda}_{a_2 a_1} \right) \mod t_H^{n-1}. \end{aligned}$$

Now the classical limit of property 1 in proposition 5.4 gives

$$\mathcal{M}_{[a_3 a_2] a_1} = Q_\Theta \tilde{Y}_{a_3 a_2 a_1} \bmod t_H^{n-1} \quad (5.15)$$

where $Y_{a_3 a_2 a_1} = \mathbb{Y}_{a_3 a_2 a_1} \big|_{\hbar=0}$. Comparing (5.10) with (5.15), we have

$$\begin{aligned} & \left(\partial_{a_3} \tilde{A}_{a_2 a_1}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \partial_{a_2} \tilde{A}_{a_3 a_1}^\gamma \right) \Theta_\gamma \\ &= Q_\Theta \left(\tilde{Y}_{a_3 a_2 a_1} - (-1)^{|\alpha_3|} \partial_{a_3} \tilde{\Lambda}_{a_2 a_1} + (-1)^{|\alpha_2|+|\alpha_3||\alpha_2|} \partial_{a_3} \tilde{\Lambda}_{a_2 a_1} \right) \bmod t_H^{n-1}. \end{aligned}$$

Applying proposition 5.1, we conclude that

$$\partial_{a_3} \tilde{A}_{a_2 a_1}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \partial_{a_2} \tilde{A}_{a_3 a_1}^\gamma = 0 \bmod t_H^{n-1}.$$

□

Corollary 5.9. *There is a choice such that*

$$\tilde{Y}_{a_3 a_2 a_1} = (-1)^{|\alpha_3|} \partial_{a_3} \tilde{\Lambda}_{a_2 a_1} - (-1)^{|\alpha_2|+|\alpha_3||\alpha_2|} \partial_{a_3} \tilde{\Lambda}_{a_2 a_1} \bmod t_H^{n-1}.$$

5.2.3. Definition of $\tilde{\Theta} \bmod t_H^{n+2}$. Now we are about to take the final step, which requires one more proposition, corresponding to the quantum gauge in $\mathbf{P}(n+1)$:

Proposition 5.5. *There is a $\tilde{\Lambda}_{a_2 a_1} \in \mathcal{C}[[t_H]]^{|\alpha_2|+|\alpha_1|-1} \bmod t_H^n$ such that*

1. $\tilde{\Lambda}_{a_2 a_1} \big|_{\hbar=0} = \tilde{\Lambda}_{a_2 a_1},$
2. $\tilde{\Lambda}_{a_2 a_1} = \Lambda_{a_2 a_1} \bmod t_H^{n-1},$
3. $\tilde{\Lambda}_{a_2 a_1} = (-1)^{|\alpha_2||\alpha_1|} \tilde{\Lambda}_{a_1 a_2},$
4. $\tilde{\Lambda}_{a_2 0} = 0,$
5. $\tilde{Y}_{a_3 a_2 a_1} = \partial_{a_3} \tilde{\Lambda}_{a_2 a_1} \bmod t_H^{n-1}.$

Proof. From the condition in proposition 5.4 that

$$\tilde{Y}_{a_3 a_2 a_1} = \partial_{a_3} \Lambda_{a_2 a_1} \bmod t_H^{n-2},$$

we are looking for a $\tilde{\Lambda}_{a_2 a_1}^{[n-1]}$ satisfying

- (a). $\tilde{\Lambda}_{a_2 a_1}^{[n-1]} \big|_{\hbar} = \tilde{\Lambda}_{a_2 a_1}^{[n-1]}$
- (b). $\tilde{\Lambda}_{a_2 a_1}^{[n-1]} = (-1)^{|\alpha_2||\alpha_1|} \tilde{\Lambda}_{a_1 a_2}^{[n-1]},$

$$(c). \tilde{\Lambda}_{a_2 0}^{[n-1]} = 0,$$

$$(d). \tilde{Y}_{a_3 a_2 a_1}^{[n-2]} = \partial_{a_3} \tilde{\Lambda}_{a_2 a_1}^{[n-1]}.$$

Assuming the above conditions, we set $\tilde{\Lambda}_{a_2 a_1} = \Lambda_{a_2 a_1} + \tilde{\Lambda}_{a_2 a_1}^{[n-1]}$. Then all the properties in the proposition are satisfied.

Consider

$$\mathbb{C}_{a_2 a_1} := \Theta_{a_2} \cdot \Theta_{a_1} - A_{a_2 a_1} \gamma \Theta_\gamma - (\Theta, \Lambda_{a_2 a_1})_{\hbar} \mod t_H^n. \quad (5.16)$$

Note that the classical limit of $\mathbb{C}_{a_2 a_1}$ is

$$\mathbb{C}_{a_2 a_1} \Big|_{\hbar=0} = \Theta_{a_2} \cdot \Theta_{a_1} - A_{a_2 a_1} \gamma \Theta_\gamma - (\Theta, \Lambda_{a_2 a_1}) \mod t_H^n.$$

Comparing above with $\Theta_{a_2} \cdot \Theta_{a_1} = \tilde{A}_{a_2 a_1} \gamma \Theta_\gamma + Q \tilde{\Lambda}_{a_2 a_1} + (\Theta, \Lambda_{a_2 a_1}) \mod t_H^n$, we have

$$\mathbb{C}_{a_2 a_1} \Big|_{\hbar=0} = \tilde{A}_{a_2 a_1}^{[n-1]} \gamma \Theta_\gamma + Q \tilde{\Lambda}_{a_2 a_1} \mod t_H^n. \quad (5.17)$$

Note also that

$$\begin{aligned} \partial_{[a_3} \mathbb{C}_{a_2] a_1} &= \mathbb{M}_{[a_3 a_2] a_1} - (\Theta, \partial_{\tilde{a}_3} \Lambda_{a_2] a_1})_{\hbar} \\ &= \mathbf{K} \tilde{Y}_{a_3 a_2 a_1} + (\Theta, \tilde{Y}_{a_3 a_2 a_1} - \partial_{\tilde{a}_3} \Lambda_{a_2] a_1})_{\hbar} \\ &\mod t_H^{n-1} \end{aligned}$$

where we have used $\mathbb{M}_{[a_3 a_2] a_1} = \mathbf{K} \tilde{Y}_{a_3 a_2 a_1} + (\Theta, \tilde{Y}_{a_3 a_2 a_1}) \mod t_H^{n-1}$ (property 1 in proposition 5.4). It follows that

$$\partial_{[a_3} \mathbb{C}_{a_2] a_1} = \mathbf{K} \tilde{Y}_{a_3 a_2 a_1} \mod t_H^{n-1}, \quad (5.18)$$

since $\tilde{Y}_{a_3 a_2 a_1} - \partial_{[\tilde{a}_3} \Lambda_{a_2] a_1} = 0 \mod t_H^{n-2}$ and $\Theta \Big|_{t_H=0} = 0$.

Now consider the word-length $(n-1)$ part $\mathbb{C}_{a_2 a_1}^{[n-1]}$ of $\mathbb{C}_{a_2 a_1}$:

$$\mathbb{C}_{a_2 a_1}^{[n-1]} = \sum_{k=0}^{n-1} \Theta_{a_2}^{[k]} \cdot \Theta_{a_1}^{[n-1-k]} - \sum_{k=0}^{n-2} A_{a_2 a_1}^{[k] \gamma} \Theta_{\gamma}^{[n-1-k]} - \sum_{k=0}^{n-2} (\Theta^{[n-1-k]}, \Lambda_{a_2 a_1}^{[k]})_{\hbar}. \quad (5.19)$$

We note, from (5.17) and (5.18), that

$$\begin{aligned} \mathbb{C}_{a_2 a_1}^{[n-1]} \Big|_{\hbar=0} &= \tilde{A}_{a_2 a_1}^{[n-1]} \gamma \Theta_\gamma + Q \tilde{\Lambda}_{a_2 a_1}^{[n-1]}, \\ \partial_{[a_3} \mathbb{C}_{a_2] a_1}^{[n-1]} &= \mathbf{K} \tilde{Y}_{a_3 a_2 a_1}^{[n-2]}. \end{aligned} \quad (5.20)$$

Hence $\tilde{Y}_{a_3 a_2 a_1}^{[n-2]} - \partial_{[\tilde{a}_3} \tilde{\Lambda}_{a_2] a_1}^{[n-1]}$ is divisible by \hbar .

Now we consider the following expansions:

$$\begin{aligned}\tilde{\mathbb{Y}}_{\alpha_3\alpha_2\alpha_1}^{[n-2]} &= \frac{1}{(n-2)!} t^{\tilde{\alpha}_4} \dots t^{\tilde{\alpha}_{n+1}} \tilde{\xi}_{\alpha_{n+1} \dots \alpha_4 \alpha_3 \alpha_2 \alpha_1} \text{ where } \xi_{\alpha_{n+1} \dots \alpha_1} \in \mathcal{C}[[\hbar]]^{|\alpha_1| + \dots + |\alpha_{n+1}| - 1}, \\ \tilde{\Lambda}_{\alpha_2\alpha_1}^{[n-1]} &= \frac{1}{(n-1)!} t^{\tilde{\alpha}_3} \dots t^{\tilde{\alpha}_{n+1}} \tilde{\lambda}_{\alpha_{n+1} \alpha_n \dots \alpha_3 \alpha_2 \alpha_1}^\gamma \text{ where } \tilde{\lambda}_{\alpha_{n+1} \dots \alpha_1} \in \mathcal{C}^{|\alpha_1| + \dots + |\alpha_{n+1}| - 1},\end{aligned}$$

such that

$$\partial_{[\tilde{\alpha}_3]} \tilde{\Lambda}_{\alpha_2\alpha_1}^{[n-1]} = \frac{1}{(n-2)!} t^{\tilde{\alpha}_4} \dots t^{\tilde{\alpha}_{n+1}} \left(\tilde{\lambda}_{\alpha_{n+1} \alpha_n \dots \alpha_4 \alpha_3 \alpha_2 \alpha_1}^\gamma - (-1)^{|\alpha_3||\alpha_2|} \tilde{\lambda}_{\alpha_{n+1} \alpha_n \dots \alpha_4 \alpha_2 \alpha_3 \alpha_1}^\gamma \right).$$

We can define $\tilde{\mathfrak{H}}_{\alpha_{n+1} \alpha_n \dots \alpha_4 \alpha_3 \alpha_2 \alpha_1} \in \mathcal{C}[[\hbar]]^{|\alpha_1| + \dots + |\alpha_{n+1}| - 1}$ as follows:

$$\hbar \tilde{\mathfrak{H}}_{\alpha_{n+1} \alpha_n \dots \alpha_4 \alpha_3 \alpha_2 \alpha_1} = \tilde{\xi}_{\alpha_{n+1} \alpha_n \dots \alpha_4 \alpha_3 \alpha_2 \alpha_1} - \left(\lambda_{\alpha_{n+1} \alpha_n \dots \alpha_4 \alpha_3 \alpha_2 \alpha_1} - (-1)^{|\alpha_3||\alpha_2|} \tilde{\lambda}_{\alpha_{n+1} \alpha_n \dots \alpha_4 \alpha_2 \alpha_3 \alpha_1}^\gamma \right),$$

since $\tilde{\mathbb{Y}}_{\alpha_3\alpha_2\alpha_1}^{[n-2]} - \partial_{[\tilde{\alpha}_3]} \tilde{\Lambda}_{\alpha_2\alpha_1}^{[n-1]}$ is divisible by \hbar . Then

$$\tilde{\mathbb{Y}}_{\gamma\beta\alpha}^{[n-2]} = \partial_{[\tilde{\gamma}]} \tilde{\Lambda}_{\beta\alpha}^{[n-1]} - \frac{\hbar}{(n-2)!} t^{\tilde{\rho}_2} \dots t^{\tilde{\rho}_{n-1}} \tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_2 \gamma \beta \alpha}.$$

Note also that

$$\begin{aligned}\tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_2 \gamma \beta \alpha} - (-1)^{|\gamma||\beta|} \tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_2 \beta \gamma \alpha} &= 2 \tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_2 \gamma \beta \alpha}, \\ (-1)^{|\beta||\alpha|} \tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_2 \gamma \alpha \beta} - (-1)^{|\gamma|(|\beta| + |\alpha|)} \tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_2 \beta \alpha \gamma} &= \tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_2 \gamma \beta \alpha}.\end{aligned}$$

Finally, we define

$$\tilde{\Lambda}_{\beta\alpha}^{[n-1]} := \tilde{\Lambda}_{\beta\alpha}^{[n-1]} - \frac{\hbar}{3(n-1)!} t^{\tilde{\rho}_1} \dots t^{\tilde{\rho}_{n-1}} \left(\tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_1 \beta \alpha} + (-1)^{|\beta||\alpha|} \tilde{\mathfrak{H}}_{\rho_{n-1} \dots \rho_1 \alpha \beta} \right)$$

Then, we have

$$\begin{aligned}(a) \quad \tilde{\Lambda}_{\alpha_2\alpha_1}^{[n-1]} \Big|_{\hbar} &= \tilde{\Lambda}_{\alpha_2\alpha_1}^{[n-1]}, \\ (b) \quad \tilde{\Lambda}_{\alpha_2\alpha_1}^{[n-1]} &= (-1)^{|\alpha_2||\alpha_1|} \tilde{\Lambda}_{\alpha_1\alpha_2}^{[n-1]}, \\ (c) \quad \tilde{\Lambda}_{\alpha_2 0}^{[n-1]} &= 0.\end{aligned}$$

Hence all that is remain is to check that $\tilde{\mathbb{Y}}_{\alpha_3\alpha_2\alpha_1}^{[n-2]} = \partial_{[\alpha_3}\tilde{\Lambda}_{\alpha_2]\alpha_1}^{[n-1]}$.

$$\begin{aligned}
\partial_{[\tilde{\gamma}}\tilde{\Lambda}_{\beta]a}^{[n-1]} &= \partial_{[\tilde{\gamma}}\tilde{\Lambda}_{\beta]a}^{[n-1]} - \frac{\hbar}{3} \frac{1}{(n-2)!} t^{\tilde{\rho}_2} \dots t^{\tilde{\rho}_{n-1}} \left(\tilde{\eta}_{\rho_{n-1} \dots \rho_2 \gamma \beta a} - (-1)^{|\gamma||\beta|} \tilde{\eta}_{\rho_{n-1} \dots \rho_2 \beta \gamma a} \right) \\
&\quad - \frac{\hbar}{3} \frac{1}{(n-2)!} t^{\tilde{\rho}_2} \dots t^{\tilde{\rho}_{n-1}} \left((-1)^{|\beta||a|} \tilde{\eta}_{\rho_{n-1} \dots \rho_2 \gamma a \beta} - (-1)^{|\gamma|(|\beta|+|a|)} \tilde{\eta}_{\rho_{n-1} \dots \rho_2 \beta a \gamma} \right) \\
&= \partial_{[\tilde{\gamma}}\tilde{\Lambda}_{\beta]a}^{[n-1]} - \frac{2\hbar}{3} \frac{1}{(n-2)!} t^{\tilde{\rho}_2} \dots t^{\tilde{\rho}_{n-1}} \tilde{\eta}_{\rho_{n-1} \dots \rho_2 \gamma \beta a} \\
&\quad - \frac{\hbar}{3} \frac{1}{(n-2)!} t^{\tilde{\rho}_2} \dots t^{\tilde{\rho}_{n-1}} \tilde{\eta}_{\rho_{n-1} \dots \rho_2 \gamma \beta a} \\
&= \partial_{[\tilde{\gamma}}\tilde{\Lambda}_{\beta]a}^{[n-1]} - \frac{\hbar}{(n-2)!} t^{\tilde{\rho}_2} \dots t^{\tilde{\rho}_{n-1}} \tilde{\eta}_{\rho_{n-1} \dots \rho_2 \gamma \beta a} \\
&= \tilde{\mathbb{Y}}_{\gamma\beta a}^{[n-1]}.
\end{aligned}$$

□.

Corollary 5.10. *Let*

$$\tilde{\mathbf{L}}_{\alpha_2\alpha_1} := \Theta_{\alpha_2} \cdot \Theta_{\alpha_1} - \tilde{A}_{\alpha_2\alpha_1}{}^\gamma \Theta_\gamma - \mathbf{K} \tilde{\Lambda}_{\alpha_2\alpha_1} - (\Theta, \tilde{\Lambda}_{\alpha_2\alpha_1})_{\hbar} \mod t_H^n.$$

Then

1. $\tilde{\mathbf{L}}_{\alpha_2\alpha_1}$ is divisible by \hbar ,
2. $\tilde{\mathbf{L}}_{\alpha_2\alpha_1} = (-1)^{|\alpha_2||\alpha_1|} \tilde{\mathbf{L}}_{\alpha_1\alpha_2} \mod t_H^n$,
3. $\tilde{\mathbf{L}}_{\alpha_2 0} = 0 \mod t_H^n$,
4. $\partial_{\alpha_3} \tilde{\mathbf{L}}_{\alpha_2\alpha_1} = (-1)^{|\alpha_3||\alpha_2|} \partial_{\alpha_2} \tilde{\mathbf{L}}_{\alpha_3\alpha_1} \mod t_H^{n-1}$

Proof. Straightforward to check in order:

1. The classical limit of $\tilde{\mathbf{L}}_{\alpha_2\alpha_1}$ vanishes due to the relation (5.6).
2. Use the relations $\tilde{A}_{\alpha_2\alpha_1}{}^\gamma = (-1)^{|\alpha_2||\alpha_1|} \tilde{A}_{\alpha_1\alpha_2} \mod t_H^n$ and $\tilde{\Lambda}_{\alpha_2\alpha_1} = (-1)^{|\alpha_2||\alpha_1|} \tilde{\Lambda}_{\alpha_1\alpha_2} \mod t_H^n$, as well as supercommutativity of the product .
3. Use the unities that $\Theta_0 = 1$ and $\tilde{A}_{0\beta}{}^\gamma = \delta_\beta{}^\gamma \mod t_H^n$ as well as the relation $\tilde{\Lambda}_{\alpha_2 0} = 0$.
4. By applying ∂_{α_3} we have

$$\partial_{\alpha_3} \tilde{\mathbf{L}}_{\alpha_2\alpha_1} = \mathbb{M}_{\alpha_3\alpha_2\alpha_1} - \partial_{\alpha_3} \tilde{A}_{\alpha_2\alpha_1}{}^\gamma \Theta_\gamma - \mathbf{K}_\Theta \partial_{\alpha_3} \tilde{\Lambda}_{\alpha_2\alpha_1} \mod t_H^{n-1}.$$

Hence

$$\begin{aligned}
& \partial_{\alpha_3} \tilde{\mathbf{L}}_{\alpha_2 \alpha_1} - (-1)^{|\alpha_3||\alpha_2|} \partial_{\alpha_2} \tilde{\mathbf{L}}_{\alpha_3 \alpha_1} \\
&= \mathbb{M}_{[\alpha_3 \alpha_2] \alpha_1} - \mathbf{K}_{\Theta} \partial_{[\tilde{\alpha}_3 \tilde{\Lambda}_{\alpha_2}] \alpha_1} \\
&= -\mathbf{K}_{\Theta} \left(\mathbf{Y}_{\alpha_3 \alpha_2 \alpha_1} - \partial_{[\tilde{\alpha}_3 \tilde{\Lambda}_{\alpha_2}] \alpha_1} \right) \\
&= 0,
\end{aligned}$$

where we have used the potentiality $\partial_{\alpha_3} \tilde{A}_{\alpha_2 \alpha_1} \gamma - (-1)^{|\alpha_3||\alpha_2|} \partial_{\alpha_2} \tilde{A}_{\alpha_3 \alpha_1} \gamma = 0 \mod t_H^{n-1}$ for the 1st equality and property 5 in proposition 5.5. \square

From the quantum master equation in $\mathbf{P}(n)$, we already know that

$$\tilde{\mathbf{L}}_{\alpha_2 \alpha_1} = \hbar \partial_{\alpha_2} \partial_{\alpha_1} \Theta \mod t_H^{n-1},$$

so that the only new piece of information in the expression $\tilde{\mathbf{L}}_{\alpha_2 \alpha_1}$ is its component $\tilde{\mathbf{L}}_{\alpha_2 \alpha_1}^{[n-1]}$ with the word-length $n-1$ in t_H :

$$\begin{aligned}
\tilde{\mathbf{L}}_{\alpha_2 \alpha_1}^{[n-1]} &= \sum_{k=0}^{n-1} \Theta_{\alpha_2}^{[k]} \cdot \Theta_{\alpha_1}^{[n-1-k]} - \sum_{k=0}^{n-2} \left(A_{\alpha\beta}^{[k]\gamma} \Theta_{\gamma}^{[n-1-k]} + \left(\Theta^{[n-1-k]}, \Lambda_{\alpha_2 \alpha_1}^{[k]} \right)_{\hbar} \right) \\
&\quad - \tilde{A}_{\alpha_2 \alpha_1}^{[n-1]\gamma} \Theta_{\gamma}^{[0]} - \mathbf{K} \tilde{\Lambda}_{\alpha_2 \alpha_1}^{[n-1]}.
\end{aligned} \tag{5.21}$$

Set $\hbar \tilde{\Theta}^{[n+1]} = \frac{1}{n(n+1)} t^{\alpha_1} t^{\alpha_2} \tilde{\mathbf{L}}_{\alpha_2 \alpha_1}^{[n-1]}$ and define

$$\tilde{\Theta} := \Theta + \tilde{\Theta}^{[n+1]} = \Theta^{[1]} + \dots + \Theta^{[n]} + \tilde{\Theta}^{[n+1]}$$

so that $\tilde{\Theta} = \Theta \mod t_H^{n+1}$ and $\partial_{\alpha} \tilde{\Theta} = \Theta_{\alpha} \mod t_H^n$. Then the following lemma finishes our construction $\mathbf{P}(n+1)$ out of the assumed $\mathbf{P}(n)$.

Lemma 5.3. $\tilde{\Theta}$ satisfies

1. the $\mod t_H^n$ quantum master equation:

$$\hbar \partial_{\alpha_2} \partial_{\alpha_1} \tilde{\Theta} = \partial_{\alpha_2} \tilde{\Theta} \cdot \partial_{\alpha_1} \tilde{\Theta} - \tilde{A}_{\alpha_2 \alpha_1} \gamma \partial_{\gamma} \tilde{\Theta} - \mathbf{K} \tilde{\Lambda}_{\alpha_2 \alpha_1} - \left(\tilde{\Theta}, \tilde{\Lambda}_{\alpha_2 \alpha_1} \right)_{\hbar} \mod t_H^n,$$

2. the quantum unity $\partial_0 \tilde{\Theta} = 1 \mod t_H^{n+1}$,

3. the $\mod t_H^{n+2}$ quantum descendant equation: $\mathbf{K} \tilde{\Theta} + \frac{1}{2} (\tilde{\Theta}, \tilde{\Theta})_{\hbar} = 0 \mod t_H^{n+2}$.

Proof. In order.

1. It is sufficient to show that $\hbar \partial_{\alpha_2} \partial_{\alpha_1} \tilde{\Theta} = \tilde{\mathbf{L}}_{\alpha_2 \alpha_1}^{[n-1]}$, which is combined with the quantum master equation $\text{mod } t_H^{n-1}$ in $\mathbf{P}(n)$ to have the $\text{mod } t_H^n$ equation. From $\tilde{\Theta}^{[n+1]} = \frac{1}{n(n+1)} t^\beta t^\gamma \tilde{\mathbf{L}}_{\gamma\beta}^{[n-1]}$, we have

$$\begin{aligned} \hbar \partial_{\alpha_1} \tilde{\Theta}^{[n+1]} &= \frac{2}{n(n+1)} t^\gamma \tilde{\mathbf{L}}_{\gamma \alpha_1}^{[n-1]} + \frac{1}{n(n+1)} (-1)^{(|\mu|+|v|)|\alpha_1|} t^\nu t^\mu \partial_{\alpha_1} \tilde{\mathbf{L}}_{\mu\nu}^{[n-1]} \\ &= \frac{2}{n(n+1)} t^\gamma \tilde{\mathbf{L}}_{\gamma \alpha_1}^{[n-1]} + \frac{1}{n(n+1)} t^\nu t^\mu \left((-1)^{|v||\alpha_1|} \partial_\mu \tilde{\mathbf{L}}_{\alpha_1 \nu}^{[n-1]} + (-1)^{(|\mu|+|v|)|\alpha_1|} \partial_{[\alpha_1} \tilde{\mathbf{L}}_{\mu] \nu}^{[n-1]} \right) \\ &= \frac{2}{n(n+1)} t^\gamma \tilde{\mathbf{L}}_{\gamma \alpha_1}^{[n-1]} + \frac{1}{n(n+1)} t^\nu t^\mu \partial_\mu \tilde{\mathbf{L}}_{\nu \alpha_1}^{[n-1]} + \frac{1}{n(n+1)} (-1)^{(|\mu|+|v|)|\alpha_1|} t^\nu t^\mu \partial_{[\alpha_1} \tilde{\mathbf{L}}_{\mu] \nu}^{[n-1]} \\ &= \frac{1}{n} t^\gamma \tilde{\mathbf{L}}_{\gamma \beta}^{[n-1]} + \frac{1}{n(n+1)} (-1)^{(|\mu|+|v|)|\alpha_1|} t^\nu t^\mu \partial_{[\alpha_1} \tilde{\mathbf{L}}_{\mu] \nu}^{[n-1]}, \end{aligned}$$

where we have used $\tilde{\mathbf{L}}_{\alpha_2 \alpha_1}^{[n-1]} = (-1)^{|\alpha_2||\alpha_1|} \tilde{\mathbf{L}}_{\alpha_1 \alpha_2}^{[n-1]}$ for the 1st and the 3rd equalities and $t^\mu \partial_\mu \mathbf{L}_{\nu \alpha_1}^{[n-1]} = (n-1) \mathbf{L}_{\nu \alpha_1}^{[n-1]}$ for the last equality. It follows that $\partial_{\alpha_1} \tilde{\Theta}^{[n+1]} = \frac{1}{n} t^\gamma \tilde{\mathbf{L}}_{\gamma \alpha_1}^{[n-1]}$ since $\partial_{[\alpha_1} \mathbf{L}_{\mu] \nu}^{[n-1]} = 0$. The similar computation show that $\hbar \partial_{\alpha_2} \partial_{\alpha_1} \tilde{\Theta} = \tilde{\mathbf{L}}_{\alpha_2 \alpha_1}^{[n-1]}$.

2. The quantum unity $\partial_0 \tilde{\Theta} = 1 \text{ mod } t_H^{n+1}$ follows from $\partial_0 \tilde{\Theta} = \partial_0 \Theta = 1 \text{ mod } t_H^n$ and $\hbar \partial_0 \tilde{\Theta}^{[n+1]} = \frac{1}{n} t^\gamma \tilde{\mathbf{L}}_{\gamma 0}^{[n-1]} = 0$.

3. In $\mathbf{P}(n)$ we have assumed that $\mathbf{K}\Theta + \frac{1}{2}(\Theta, \Theta)_\hbar = \mathbf{K}\tilde{\Theta} + \frac{1}{2}(\tilde{\Theta}, \tilde{\Theta})_\hbar = 0 \text{ mod } t_H^{n+1}$. So it is suffice to show that

$$\mathbf{K}\tilde{\Theta}^{[n+1]} + \frac{1}{2} \sum_{k=1}^n \left(\Theta^{[n+1-k]}, \Theta^{[k]} \right)_\hbar = 0. \quad (5.22)$$

Note that the quantum master equation in 1 can be written as follows:

$$\hbar \partial_{\alpha_2} \partial_{\alpha_1} \tilde{\Theta} = \partial_{\alpha_2} \Theta \cdot \partial_{\alpha_1} \Theta - \tilde{A}_{\alpha_2 \alpha_1}{}^\gamma \partial_\gamma \Theta - \mathbf{K} \tilde{\Lambda}_{\alpha_2 \alpha_1} - (\Theta, \tilde{\Lambda}_{\alpha_2 \alpha_1})_\hbar \text{ mod } t_H^n.$$

Applying $\mathbf{K}\Theta$ to the above, we have

$$\mathbf{K} \partial_{\alpha_2} \partial_{\alpha_1} \tilde{\Theta} + (\Theta, \partial_{\alpha_2} \partial_{\alpha_1} \tilde{\Theta})_\hbar = -\hbar (-1)^{|\alpha_2|} (\partial_{\alpha_2} \Theta, \partial_{\alpha_1} \Theta)_\hbar \text{ mod } t_H^n.$$

Now consider the word length $(n-1)$ part of the above identity:

$$\mathbf{K} (\partial_{\alpha_2} \partial_{\alpha_1} \tilde{\Theta}^{[n+1]}) + \sum_{k=1}^n (\Theta^{[n+1-k]}, \partial_{\alpha_2} \partial_{\alpha_1} \Theta^{[k]})_\hbar = -(-1)^{|\alpha_2|} \sum_{k=1}^n (\partial_{\alpha_2} \Theta^{[n+1-k]}, \partial_{\alpha_1} \Theta^{[k]})_\hbar,$$

and multiply $(-1)^{|\alpha_2|+|\alpha_1|} t^{\alpha_1} t^{\alpha_2}$ to sum over the repeated indices such that

$$\begin{aligned} \mathbf{K} (t^{\alpha_1} t^{\alpha_2} \partial_{\alpha_2} \partial_{\alpha_1} \tilde{\Theta}^{[n+1]}) + \sum_{k=1}^n (\Theta^{[n+1-k]}, t^{\alpha_1} t^{\alpha_2} \partial_{\alpha_2} \partial_{\alpha_1} \Theta^{[k]})_\hbar \\ = - \sum_{k=1}^n (t^{\alpha_2} \partial_{\alpha_2} \Theta^{[n+1-k]}, t^{\alpha_1} \partial_{\alpha_1} \Theta^{[k]})_\hbar. \end{aligned}$$

Then, from the homogeneity of $\Theta^{[k]}$ such that $t^\alpha \partial_\alpha \Theta^{[k]} = k \Theta^{[k]}$, we obtain that

$$n(n+1)\mathbf{K}\tilde{\Theta}^{[n+1]} + \sum_{k=1}^n k(k-1) \left(\Theta^{[n+1-k]}, \Theta^{[k]} \right)_\hbar = - \sum_{k=1}^n (n+1-k)k \left(\Theta^{[n+1-k]}, \Theta^{[k]} \right)_\hbar,$$

which is simplified as follows:

$$n(n+1)\mathbf{K}\tilde{\Theta}^{[n+1]} + \sum_{k=1}^n k \left(\Theta^{[n+1-k]}, \Theta^{[k]} \right)_\hbar = 0.$$

Using the following re-summation:

$$\begin{aligned} \sum_{k=1}^n k \left(\Theta^{[n+1-k]}, \Theta^{[k]} \right)_\hbar &= \sum_{k=1}^n (n+1-k) \left(\Theta^k, \Theta^{[n+1-k]} \right)_\hbar \\ &= \sum_{k=1}^n (n+1-k) \left(\Theta^{[n+1-k]}, \Theta^{[k]} \right)_\hbar, \end{aligned}$$

we have

$$(n+1)\mathbf{K}\tilde{\Theta}^{[n+1]} + \frac{(n+1)}{2} \sum_{k=1}^n \left(\Theta^{[n+1-k]}, \Theta^{[k]} \right)_\hbar = 0,$$

which is equivalent the relation (5.22). \square

Finally take $n \rightarrow \infty$ and we are done.

A. Appendix

The purpose of this appendix is to prove proposition 2.1 and 2.2, which compare $(\mathcal{C}[[\hbar]], \mathbf{K})$ and $(H[[\hbar]], \mathbf{\kappa})$ both as cochain complexes over $\mathbb{k}[[\hbar]]$.

Recall that a quantization map $\mathbf{f} = f + \hbar f^{(1)} + \dots$ is a ghost number preserving $\mathbb{k}[[\hbar]]$ -linear map on $H[[\hbar]]$ to $\mathcal{C}[[\hbar]]$ satisfying $\mathbf{K}\mathbf{f} = \mathbf{f}\mathbf{\kappa}$ such that $Qf = 0$, $f(0) = 0$, and f induces the identity map on H , i.e, the Q -cohomology class of $f(x)$ is x for all $x \in H$. Let $\gamma \in \text{Ker } Q \cap \mathcal{C}$. We denote the Q -cohomology class of γ by $[\gamma] \in H$. Note that $f([\gamma]) = \gamma \bmod \text{Im } Q$ since f induces the identity map on H . Note also that an equality of the type $f(x) = Q\lambda$, where $x \in H^{|x|}$ and $\lambda \in \mathcal{C}^{|x|-1}$ implies that (i) $x = 0$ since $[f(x)] = [Q\lambda] = 0$ and $[f(x)] = x$, and (ii) $Q\lambda = 0$ since $f(0) = 0$.

We begin with proposition 2.1:

Proposition A.1. *Any homogeneous $\eta \in \mathcal{C}[[\hbar]]^{|\eta|}$ satisfying $\mathbf{K}\eta = 0$, can be expressed as*

$$\eta = \mathbf{f}(\mathbf{x}) + \mathbf{K}\lambda,$$

for certain pair $(\mathbf{x}, \lambda) \in H[[\hbar]]^{|\eta|} \oplus \mathcal{C}[[\hbar]]^{|\eta|-1}$ such that $\kappa\mathbf{x} = 0$. Let $(\mathbf{x}', \lambda') \in H[[\hbar]]^{|\eta|} \oplus \mathcal{C}[[\hbar]]^{|\eta|-1}$ be any other pair satisfying

$$\eta = \mathbf{f}(\mathbf{x}') + \mathbf{K}\lambda'.$$

Then there is certain pair $(\mathbf{y}, \zeta) \in H[[\hbar]]^{|\eta|-1} \oplus \mathcal{C}[[\hbar]]^{|\eta|-2}$ such that

$$\begin{aligned} \mathbf{x}' - \mathbf{x} &= -\kappa\mathbf{y}, \\ \lambda' - \lambda &= \mathbf{f}(\mathbf{y}) + \mathbf{K}\zeta. \end{aligned}$$

Consider any $\eta \in \mathcal{C}[[\hbar]]$ satisfying $\mathbf{K}\eta = 0$. We shall show that (i) there is a pair (\mathbf{x}, λ) such that $\eta = \mathbf{f}(\mathbf{x}) + \mathbf{K}\lambda$, (ii) $\kappa\mathbf{x} = 0$ and (iii) for any other (\mathbf{x}', λ') satisfying $\eta = \mathbf{f}(\mathbf{x}') + \mathbf{K}\lambda'$ there is a pair (\mathbf{y}, ζ) such that $\mathbf{x}' - \mathbf{x} = -\kappa\mathbf{y}$ and $\lambda' - \lambda = \mathbf{f}(\mathbf{y}) + \mathbf{K}\zeta$. Denote $\eta = \eta^{(0)} + \hbar\eta^{(1)} + \dots$.

We begin with proving the proposition modulo \hbar . Note that the condition $\mathbf{K}\eta = 0$ modulo \hbar is $Q\eta^{(0)} = 0$. Then,

(i) It follows that $\eta^{(0)} = f([\eta^{(0)}]) + Q\lambda^{(0)}$ for some $\lambda^{(0)} \in \mathcal{C}$. Set $x^{(0)} = [\eta^{(0)}]$, i.e.,

$$\eta^{(0)} = f(x^{(0)}) + Q\lambda^{(0)}. \quad (\text{A.23})$$

Set

$$\begin{aligned} \mathbf{x} &= x^{(0)} \bmod \hbar, \\ \lambda &= \lambda^{(0)} \bmod \hbar. \end{aligned}$$

Then the relation (A.23) is equivalent to $\eta = \mathbf{f}(\mathbf{x}) + \mathbf{K}\lambda \bmod \hbar$.

(ii) It is obvious that $\kappa\mathbf{x} = 0 \bmod \hbar$ since $\kappa = 0 \bmod \hbar$.

(iii) Let $(x'^{(0)}, \lambda'^{(0)})$ be any other pair satisfying $\eta^{(0)} = f(x'^{(0)}) + Q\lambda'^{(0)}$. Then, by a comparison with (A.23), we have

$$f(x'^{(0)} - x^{(0)}) = -Q(\lambda'^{(0)} - \lambda^{(0)}).$$

By taking the Q -cohomology class to the above, we conclude that $x'^{(0)} = x^{(0)}$ as well as $Q(\lambda'^{(0)} - \lambda^{(0)}) = 0$, which implies that there is a pair $(y^{(0)}, \zeta^{(0)})$ such that $\lambda'^{(0)} - \lambda^{(0)} = f(y^{(0)}) + Q\zeta^{(0)}$:

$$\begin{aligned} x'^{(0)} - x^{(0)} &= 0, \\ \lambda'^{(0)} - \lambda^{(0)} &= f(y^{(0)}) + Q\zeta^{(0)}. \end{aligned}$$

The above set of relations is equivalent to

$$\begin{aligned}\mathbf{x}' - \mathbf{x} &= -\mathbf{\kappa}\mathbf{y} \bmod \hbar, \\ \boldsymbol{\lambda}' - \boldsymbol{\lambda} &= \mathbf{f}(\mathbf{y}) + \mathbf{K}\boldsymbol{\zeta},\end{aligned}$$

where

$$\begin{aligned}\mathbf{x}' &= x^{(0)} \bmod \hbar, & \mathbf{y} &= y^{(0)} \bmod \hbar, \\ \boldsymbol{\lambda}' &= \lambda^{(0)} \bmod \hbar, & \boldsymbol{\zeta} &= \zeta^{(0)} \bmod \hbar.\end{aligned}$$

The above demonstration is too trivial to be an useful example, though it is a necessary step for mathematical induction. Now the next order is demonstrated for pedagogical purposes—since it has all the essential features of the proposition, some readers may read it and skip the actual proof.

Consider the proposition modulo \hbar^2 . Then the condition $\mathbf{K}\boldsymbol{\eta} = 0 \bmod \hbar^2$ is $Q\eta^{(0)} = 0$ and $Q\eta^{(1)} + K^{(1)}\eta^{(0)} = 0$, which becomes

$$Q\eta^{(1)} + K^{(1)}f(x^{(0)}) + K^{(1)}Q\lambda^{(0)} = 0,$$

after using (A.23). From $\mathbf{K}\mathbf{f} = \mathbf{f}\mathbf{\kappa}$ and $\mathbf{K}^2 = 0$, we have $K^{(1)}f = -Qf^{(1)} + f\kappa^{(1)}$ and $K^{(1)}Q = -QK^{(1)}$. Thus the above condition is equivalent to

$$Q(\eta^{(1)} - f^{(1)}(x^{(0)}) - K^{(1)}\lambda^{(0)}) = -f(\kappa^{(1)}x^{(0)}). \quad (\text{A.24})$$

By taking the Q -cohomology class to the above, we have

$$0 = \kappa^{(1)}x^{(0)}. \quad (\text{A.25})$$

Then (A.24) also implies that

$$Q(\eta^{(1)} - f^{(1)}(x^{(0)}) - K^{(1)}\lambda^{(0)}) = 0. \quad (\text{A.26})$$

(i) From (A.26), It follows that $\eta^{(1)} - f^{(1)}(x^{(0)}) - K^{(1)}\lambda^{(0)} = f(x^{(1)}) + Q\lambda^{(1)}$ for unique $x^{(1)} \in H$ and some $\lambda^{(1)} \in \mathcal{C}$. Then, together with (A.23), we have

$$\begin{aligned}\eta^{(0)} &= f(x^{(0)}) + Q\lambda^{(0)}, \\ \eta^{(1)} &= f^{(1)}(x^{(0)}) + f(x^{(1)}) + K^{(1)}\lambda^{(0)} + Q\lambda^{(1)}.\end{aligned} \quad (\text{A.27})$$

Let

$$\begin{aligned}\mathbf{x} &:= x^{(0)} + \hbar x^{(1)} \bmod \hbar^2, \\ \boldsymbol{\lambda} &:= \lambda^{(0)} + \hbar \lambda^{(1)} \bmod \hbar^2.\end{aligned}$$

Then the relation (A.27) is equivalent to $\boldsymbol{\eta} = \mathbf{f}(\mathbf{x}) + \mathbf{K}\boldsymbol{\lambda} \bmod \hbar^2$.

(ii) From (A.25), it follows $\mathbf{\kappa}\mathbf{x} = \kappa^{(1)}x^{(0)} = 0 \bmod \hbar^2$ since $\mathbf{\kappa} = 0 \bmod \hbar$.

(iii) Recall that $\lambda^{(0)}$ is defined modulo $\text{Ker } Q$ in (A.23). Let $\lambda'^{(0)} = \lambda^{(0)} + \xi^{(0)}$ for any $\xi^{(0)} \in \text{Ker } Q$. Then we also have

$$\eta^{(0)} = f(x^{(0)}) + Q\lambda'^{(0)}.$$

Repeating the same steps of deriving (A.24) to (A.26), we conclude that

$$Q(\eta^{(1)} - f^{(1)}(x^{(0)}) - K^{(1)}\lambda'^{(0)}) = 0.$$

It follows that $\eta^{(1)} - f^{(1)}(x^{(0)}) - K^{(1)}\lambda'^{(0)} = f(x'^{(1)}) + Q\lambda'^{(1)}$ for unique $x'^{(1)} \in H$ and some $\lambda'^{(1)} \in \mathcal{C}$, i.e.,

$$\eta^{(1)} = f^{(1)}(x^{(0)}) + f(x'^{(1)}) + K^{(1)}\lambda'^{(0)} + Q\lambda'^{(1)}. \quad (\text{A.28})$$

By comparing the above with (A.27) we have

$$f(x'^{(1)} - x^{(1)}) = -K^{(1)}\xi^{(0)} - Q(\lambda'^{(1)} - \lambda^{(1)}). \quad (\text{A.29})$$

Recall that $\xi^{(0)} = \lambda'^{(0)} - \lambda^{(0)}$ is an element in $\text{Ker } Q$. Hence

$$\xi^{(0)} = f(y^{(0)}) + Q\zeta^{(0)} \quad (\text{A.30})$$

for $y^{(0)} = [\xi^{(0)}] \in H$ and some $\zeta^{(0)} \in \mathcal{C}$. Recall also that $\xi^{(0)}$ can be an arbitrary element in $\text{Ker } Q$ so that $y^{(0)}$ is an arbitrary element in H accordingly. From

$$\begin{aligned} K^{(1)}\xi^{(0)} &= K^{(1)}f(y^{(0)}) + K^{(1)}Q\zeta^{(0)} \\ &= f(\kappa^{(1)}y^{(0)}) - Qf^{(1)}(y^{(0)}) - QK^{(1)}\zeta^{(0)}, \end{aligned}$$

the relation (A.29) becomes

$$f(x'^{(1)} - x^{(1)} + \kappa^{(1)}y^{(0)}) = -Q(\lambda'^{(1)} - \lambda^{(1)} - f^{(1)}(y^{(0)}) - K^{(1)}\zeta^{(0)}). \quad (\text{A.31})$$

By taking the Q -cohomology class of the above, we have

$$x'^{(1)} = x^{(1)} - \kappa^{(1)}y^{(0)}. \quad (\text{A.32})$$

Hence (A.31) also implies that

$$Q(\lambda'^{(1)} - \lambda^{(1)} - f^{(1)}(y^{(0)}) - K^{(1)}\zeta^{(0)}) = 0,$$

so that there is unique $y^{(1)} \in H$ and some $\zeta^{(1)} \in \mathcal{C}$ such that

$$\lambda'^{(1)} - \lambda^{(1)} = f^{(1)}(y^{(0)}) + f^{(1)}(y^{(1)}) + K^{(1)}\zeta^{(0)} + Q\zeta^{(1)}. \quad (\text{A.33})$$

Let $\mathbf{y} = y^{(0)} + \hbar y^{(1)} \bmod \hbar^2$. From (A.32), we conclude that the solution $\mathbf{x} = x^{(0)} + \hbar x^{(1)} \bmod \hbar^2$ is unique up to $\kappa \mathbf{y} = \kappa^{(1)}y^{(0)} \bmod \hbar^2$ for some $\mathbf{y} = y^{(0)} + \hbar y^{(1)} \bmod \hbar^2$ in $H[[\hbar]]$. Combining together (A.30) and (A.33), we also conclude that

$$\lambda' - \lambda = \mathbf{f}(\mathbf{y}) + \mathbf{K}\boldsymbol{\zeta} \bmod \hbar^2$$

where $\boldsymbol{\zeta} = \zeta^{(0)} + \hbar \zeta^{(1)} \bmod \hbar^2$.

Proof. Consider $\boldsymbol{\eta} = \eta^{(0)} + \hbar\eta^{(1)} + \dots \in \mathcal{C}[[\hbar]]$ satisfying $\mathbf{K}\boldsymbol{\eta} = 0$. Fix $n \geq 1$ and assume that

(1) there is a pair $\{\mathbf{x}, \boldsymbol{\lambda}\}$ which satisfies $\boldsymbol{\eta} = \mathbf{f}(\mathbf{x}) + \mathbf{K}\boldsymbol{\lambda} \bmod \hbar^n$, where

$$\begin{aligned}\mathbf{x} &= x^{(0)} + \hbar x^{(1)} + \dots + \hbar^{n-1} x^{(n-1)}, \\ \boldsymbol{\lambda} &= \lambda^{(0)} + \hbar \lambda^{(1)} + \dots + \hbar^{n-1} \lambda^{(n-1)},\end{aligned}$$

(2) $\boldsymbol{\kappa}\mathbf{x} = 0 \bmod \hbar^n$,

(3) for any other pair $\{\mathbf{x}', \boldsymbol{\lambda}'\}$ satisfying $\boldsymbol{\eta} = \mathbf{f}(\mathbf{x}') + \mathbf{K}\boldsymbol{\lambda}' \bmod \hbar^n$, there is a pair $\{\mathbf{y}, \boldsymbol{\zeta}\}$, where

$$\begin{aligned}\mathbf{y} &= y^{(0)} + \hbar y^{(1)} + \dots + \hbar^{n-1} y^{(n-1)}, \\ \boldsymbol{\zeta} &= \zeta^{(0)} + \hbar \zeta^{(1)} + \dots + \hbar^{n-1} \zeta^{(n-1)},\end{aligned}$$

such that

$$\begin{aligned}\mathbf{x}' - \mathbf{x} &= -\boldsymbol{\kappa}\mathbf{y} \bmod \hbar^n, \\ \boldsymbol{\lambda}' - \boldsymbol{\lambda} &= \mathbf{f}(\mathbf{y}) + \mathbf{K}\boldsymbol{\zeta} \bmod \hbar^n.\end{aligned}$$

From the given condition $\mathbf{K}\boldsymbol{\eta} = 0$, we have

$$Q\eta^{(n)} + \sum_{\ell=0}^{n-1} K^{(n-\ell)} \eta^{(\ell)} = 0. \quad (\text{A.34})$$

From assumption (1) we have, for $0 \leq \ell \leq n-1$,

$$\eta^{(\ell)} = \sum_{j=0}^{\ell} f^{(j)}(x^{(\ell-j)}) + Q\lambda^{(\ell)} + \sum_{j=1}^{\ell} K^{(j)} \lambda^{(\ell-j)}.$$

Hence (A.34) becomes

$$Q\eta^{(n)} + \sum_{\ell=0}^{n-1} \sum_{j=0}^{\ell} K^{(n-\ell)} f^{(j)}(x^{(\ell-j)}) + \sum_{\ell=0}^{n-1} \sum_{j=1}^{\ell} K^{(n-\ell)} K^{(j)} \lambda^{(\ell-j)} + \sum_{\ell=0}^{n-1} K^{(n-\ell)} Q\lambda^{(\ell)} = 0,$$

which is equivalent to

$$\begin{aligned}Q \left(\eta^{(n)} - \sum_{\ell=0}^{n-1} f^{(\ell)\gamma}(x^{(n-\ell)}) - \sum_{\ell=0}^{n-1} K^{(n-\ell)} \lambda^{(\ell)} \right) &= - \sum_{\ell=1}^{n-1} f^{(n-\ell)} \left(\sum_{j=1}^{\ell} \kappa^{(j)} x^{(\ell-j)} \right) \\ &\quad - f \left(\sum_{i=1}^n \kappa^{(i)} x^{(n-i)} \right),\end{aligned} \quad (\text{A.35})$$

where we have used $\mathbf{K}^2 = 0$ and $\mathbf{K}\mathbf{f} = \mathbf{K}\mathbf{f}$ with some resummations. From assumption (2) we have $\sum_{j=1}^{\ell} \kappa^{(j)} x^{(\ell-j)} = 0$ for all $0 \leq \ell \leq n-1$. Hence (A.35) reduces to

$$Q \left(\eta^{(n)} - \sum_{\ell=0}^{n-1} f^{(\ell)\gamma} (x^{(n-\ell)}) - \sum_{\ell=0}^{n-1} K^{(n-\ell)} \lambda^{(\ell)} \right) = -f \left(\sum_{i=1}^n \kappa^{(i)} x^{(n-i)} \right). \quad (\text{A.36})$$

Then, by taking the Q -cohomology class of the above relation, we have

$$\sum_{i=1}^n \kappa^{(i)} x^{(n-i)} = 0, \quad (\text{A.37})$$

as well as

$$Q \left(\eta^{(n)} - \sum_{\ell=0}^{n-1} f^{(\ell)\gamma} (x^{(n-\ell)}) - \sum_{\ell=0}^{n-1} K^{(n-\ell)} \lambda^{(\ell)} \right) = 0. \quad (\text{A.38})$$

It follows, from (A.38) that there is a $x^{(n)} \in H$ and some $\lambda^{(n)} \in \mathcal{C}$ such that

$$\eta^{(n)} - \sum_{\ell=0}^{n-1} f^{(\ell)\gamma} (x^{(n-\ell)}) - \sum_{\ell=0}^{n-1} K^{(n-\ell)} \lambda^{(\ell)} = f(x^{(n)}) + Q\lambda^{(n)}.$$

Equivalently, we have

$$\eta^{(n)} = \sum_{\ell=0}^n f^{(\ell)} (x^{(n-\ell)}) + \sum_{\ell=0}^{n-1} K^{(n-\ell)} \lambda^{(\ell)} + Q\lambda^{(n)}. \quad (\text{A.39})$$

Set

$$\begin{aligned} \tilde{\mathbf{x}} &= \mathbf{x} + \hbar^n x^{(n)}, \\ \tilde{\boldsymbol{\lambda}} &= \boldsymbol{\lambda} + \hbar^n \lambda^{(n)}. \end{aligned}$$

Then (A.39) together with assumption (1) is equivalent to the following:

(1). The pair $\{\tilde{\mathbf{x}}, \tilde{\boldsymbol{\lambda}}\}$ satisfies $\boldsymbol{\eta} = \mathbf{f}(\tilde{\mathbf{x}}) + \mathbf{K}\tilde{\boldsymbol{\lambda}} \bmod \hbar^{n+1}$.

The relation (A.37) together with assumption (2) is equivalent to the following:

(2). $\mathbf{K}\tilde{\mathbf{x}} = 0 \bmod \hbar^{n+1}$.

Let $\{\mathbf{x}', \boldsymbol{\lambda}'\}$ be any other pair satisfying $\boldsymbol{\eta} = \mathbf{f}(\mathbf{x}') + \mathbf{K}\boldsymbol{\lambda}' \bmod \hbar^n$ such that, by assumption (3),

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \mathbf{K}\mathbf{y}, \\ \boldsymbol{\lambda}' &= \boldsymbol{\lambda} + \mathbf{f}(\mathbf{y}) + \mathbf{K}\boldsymbol{\zeta}, \end{aligned}$$

where $\mathbf{x}' = x'^{(0)} + \hbar x'^{(1)} + \dots + \hbar^{n-1} x'^{(n-1)}$ and $\boldsymbol{\lambda}' = \lambda'^{(0)} + \hbar \lambda'^{(1)} + \dots + \hbar^{n-1} \lambda'^{(n-1)}$. Then we also have we have, for $0 \leq \ell \leq n-1$,

$$\eta^{(\ell)} = \sum_{j=0}^{\ell} f^{(j)} (x'^{(\ell-j)}) + Q\lambda'^{(\ell)} + \sum_{j=1}^{\ell} K^{(j)} \lambda'^{(\ell-j)},$$

Repeating the same steps as the derivations of (A.35) to (A.38), we can conclude that

$$Q \left(\eta^{(n)} - \sum_{\ell=0}^{n-1} f^{(\ell)} \gamma \left(x'^{(n-\ell)} \right) - \sum_{\ell=0}^{n-1} K^{(n-\ell)} \lambda'^{(\ell)} \right) = 0. \quad (\text{A.40})$$

Hence there is a $x'^{(n)} \in H$ and some $\lambda'^{(n)} \in \mathcal{C}$ such that

$$\eta^{(n)} = \sum_{\ell=0}^n f^{(\ell)} \left(x'^{(n-\ell)} \right) + \sum_{\ell=0}^{n-1} K^{(n-\ell)} \lambda'^{(\ell)} + Q \lambda'^{(n)}. \quad (\text{A.41})$$

By comparing the above with (A.39) we have

$$f \left(x'^{(n)} - x^{(n)} \right) + Q \left(\lambda'^{(n)} - \lambda^{(n)} \right) = - \sum_{\ell=0}^{n-1} f^{(n-\ell)} \left(x'^{(\ell)} - x^{(\ell)} \right) - \sum_{\ell=0}^{n-1} K^{(n-\ell)} \left(\lambda'^{(\ell)} - \lambda^{(\ell)} \right). \quad (\text{A.42})$$

The RHS of the above can be rewritten as follows:

$$RHS = f \left(\sum_{\ell=1}^n \kappa^{(\ell)} y^{(n-\ell)} \right) + Q \sum_{\ell=1}^n \left(f^{(\ell)} \left(y^{(n-\ell)} \right) + K^{(\ell)} \zeta^{(n-\ell)} \right),$$

where we have used assumption (3) in components, that is, for $0 \leq \ell \leq n-1$,

$$\begin{aligned} x'^{(\ell)} - x^{(\ell)} &= \sum_{j=1}^{\ell} \kappa^{(j)} y^{(\ell-j)}, \\ \lambda'^{(\ell)} - \lambda^{(\ell)} &= \sum_{j=0}^{\ell} f^{(j)} \left(y^{(\ell-j)} \right) + \sum_{j=0}^{\ell} K^{(j)} y^{(\ell-j)}, \end{aligned}$$

and $\mathbf{K}^2 = \mathbf{K}\mathbf{f} - \mathbf{f}\mathbf{K} = 0$. Hence (A.42) becomes

$$f \left(x'^{(n)} - x^{(n)} + \sum_{\ell=1}^n \kappa^{(\ell)} y^{(n-\ell)} \right) = -Q \left(\lambda'^{(n)} - \lambda^{(n)} - \sum_{\ell=1}^n \left(f^{(\ell)} \left(y^{(n-\ell)} \right) + K^{(\ell)} \zeta^{(n-\ell)} \right) \right). \quad (\text{A.43})$$

By taking the Q -cohomology class of the above, we have

$$x'^{(n)} - x^{(n)} = - \sum_{\ell=1}^n \kappa^{(\ell)} y^{(n-\ell)}. \quad (\text{A.44})$$

Then, (A.44) is reduced to

$$Q \left(\lambda'^{(n)} - \lambda^{(n)} - \sum_{\ell=1}^n \left(f^{(\ell)} \left(y^{(n-\ell)} \right) + K^{(\ell)} \zeta^{(n-\ell)} \right) \right) = 0.$$

It follows that there is a pair $\{y^{(n)}, \zeta^{(n)}\}$ such that

$$\lambda'^{(n)} - \lambda^{(n)} - \sum_{\ell=1}^n \left(f^{(\ell)}(y^{(n-\ell)}) + K^{(\ell)} \zeta^{(n-\ell)} \right) = f(y^{(n)}) + Q \zeta^{(n)}.$$

Equivalently

$$\lambda'^{(n)} - \lambda^{(n)} = f(y^{(n)}) + \sum_{\ell=1}^n f^{(\ell)}(y^{(n-\ell)}) + Q \zeta^{(n)} + \sum_{\ell=1}^n K^{(\ell)} \zeta^{(n-\ell)}. \quad (\text{A.45})$$

Set

$$\begin{aligned} \tilde{\mathbf{y}} &= \mathbf{y} + \hbar^n y^{(n)}, \\ \tilde{\boldsymbol{\zeta}} &= \boldsymbol{\zeta} + \hbar^n \zeta^{(n)}. \end{aligned}$$

Then the relations (A.44) and (A.45) together with assumption (3) imply the following:

(3). For the another pair $\{\tilde{\mathbf{x}}', \tilde{\boldsymbol{\lambda}}'\}$ satisfying $\boldsymbol{\eta} = \mathbf{f}(\tilde{\mathbf{x}}') + \mathbf{K} \tilde{\boldsymbol{\lambda}}' \bmod \hbar^{n+1}$, we have

$$\begin{aligned} \tilde{\mathbf{x}}' - \tilde{\mathbf{x}} &= -\boldsymbol{\kappa} \tilde{\mathbf{y}} \bmod \hbar^{n+1}, \\ \tilde{\boldsymbol{\lambda}}' - \tilde{\boldsymbol{\lambda}} &= \mathbf{f}(\tilde{\mathbf{y}}) + \mathbf{K} \tilde{\boldsymbol{\zeta}} \bmod \hbar^{n+1}. \end{aligned}$$

Hence by mathematical induction, we have the proposition. \square

Now we turn to proposition 2.2:

Proposition A.2. A pair $\{\mathbf{x}, \boldsymbol{\lambda}\} \in H[[\hbar]]^{|\mathbf{x}|} \oplus \mathcal{C}[[\hbar]]^{|\mathbf{x}|-1}$ satisfies

$$\mathbf{f}(\mathbf{x}) = \mathbf{K} \boldsymbol{\lambda}$$

if and only if there is a pair $\{\mathbf{y}, \boldsymbol{\zeta}\} \in H[[\hbar]]^{|\mathbf{x}|-1} \oplus \mathcal{C}[[\hbar]]^{|\mathbf{x}|-2}$ such that

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\kappa} \mathbf{y}, \\ \boldsymbol{\lambda} &= \mathbf{f}(\mathbf{y}) + \mathbf{K} \boldsymbol{\zeta} \end{aligned}$$

Proof. Assume that $\mathbf{x} = \boldsymbol{\kappa} \mathbf{y}$ and $\boldsymbol{\lambda} = \mathbf{f}(\mathbf{y}) + \mathbf{K} \boldsymbol{\zeta}$. Then $\mathbf{K} \boldsymbol{\lambda} = \mathbf{K} \mathbf{f}(\mathbf{y}) = \mathbf{f}(\boldsymbol{\kappa} \mathbf{y})$. Hence $\mathbf{K} \boldsymbol{\lambda} = \mathbf{f}(\mathbf{x})$. It remains to show that there is a pair $\{\mathbf{y}, \boldsymbol{\zeta}\}$ such that $\mathbf{x} = \boldsymbol{\kappa} \mathbf{y}$ and $\boldsymbol{\lambda} = \mathbf{f}(\mathbf{y}) + \mathbf{K} \boldsymbol{\zeta}$ if $\mathbf{f}(\mathbf{x}) = \mathbf{K} \boldsymbol{\lambda}$.

– The condition $\mathbf{f}(\mathbf{x}) = \mathbf{K} \boldsymbol{\lambda}$ modulo \hbar is

$$f(x^{(0)}) = Q \lambda^{(0)}. \quad (\text{A.46})$$

It follows that $x^{(0)} = 0$ and $Q\lambda^{(0)} = 0$, which implies that there is a pair $\{y^{(0)}, \zeta^{(0)}\}$ such that $\lambda^{(0)} = f(y^{(0)}) + Q\zeta^{(0)}$:

$$\begin{aligned} x^{(0)} &= 0, \\ \lambda^{(0)} &= f(y^{(0)}) + Q\zeta^{(0)}, \end{aligned} \tag{A.47}$$

Set $\mathbf{y} = y^{(0)} \bmod \hbar$ and $\boldsymbol{\zeta} = \zeta^{(0)} \bmod \hbar$. Then (A.47) is equivalent to

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\kappa} \mathbf{y} \bmod \hbar, \\ \boldsymbol{\lambda} &= \mathbf{f}(\mathbf{y}) + \mathbf{K} \boldsymbol{\zeta} \bmod \hbar \end{aligned} \tag{A.48}$$

– The condition $\mathbf{f}(\mathbf{x}) = \mathbf{K} \boldsymbol{\lambda}$ modulo \hbar^2 is equivalent to (A.46) and

$$f(x^{(1)}) + f^{(1)}(x^{(0)}) = K^{(1)}\lambda^{(0)} + Q\lambda^{(1)}. \tag{A.49}$$

From (A.47), we have $f^{(1)}(x^{(0)}) = 0$ and

$$\begin{aligned} K^{(1)}\lambda^{(0)} &= K^{(1)}f(y^{(0)}) + K^{(1)}Q\zeta^{(0)} \\ &= f(\kappa^{(1)}y^{(0)}) - Q(f^{(1)}(y^{(0)}) + K^{(1)}\zeta^{(0)}).. \end{aligned}$$

Thus, (A.49) becomes

$$f(x^{(1)} - \kappa^{(1)}y^{(0)}) = Q(\lambda^{(1)} - f^{(1)}(y^{(0)}) - K^{(1)}\zeta^{(0)}).$$

It follows that

$$x^{(1)} = \kappa^{(1)}y^{(0)} \tag{A.50}$$

and $Q(\lambda^{(1)} - f^{(1)}(y^{(0)}) - K^{(1)}\zeta^{(0)}) = 0$, which implies that there is a pair $\{y^{(1)}, \zeta^{(1)}\}$ such that $\lambda^{(1)} - f^{(1)}(y^{(0)}) - K^{(1)}\zeta^{(0)} = f(y^{(1)}) + Q\zeta^{(1)}$, i.e.,

$$\lambda^{(1)} = f^{(1)}(y^{(0)}) + f(y^{(1)}) + K^{(1)}\zeta^{(0)} + Q\zeta^{(1)}. \tag{A.51}$$

Set $\mathbf{y} = y^{(0)} + \hbar y^{(1)} \bmod \hbar^2$ and $\boldsymbol{\zeta} = \zeta^{(0)} + \hbar \zeta^{(1)} \bmod \hbar^2$. Then (A.50) and (A.51) together with (A.48) are equivalent to

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\kappa} \mathbf{y} \bmod \hbar^2, \\ \boldsymbol{\lambda} &= \mathbf{f}(\mathbf{y}) + \mathbf{K} \boldsymbol{\zeta} \bmod \hbar^2. \end{aligned} \tag{A.52}$$

– (Assumption): Fix $n > 2$ and assume that there is a pair $\{\mathbf{y}, \boldsymbol{\zeta}\}$, where

$$\begin{aligned} \mathbf{y} &= y^{(0)} + \hbar y^{(1)} + \dots + \hbar^{n-1} y^{(n-1)}, \\ \boldsymbol{\zeta} &= \zeta^{(0)} + \hbar \zeta^{(1)} + \dots + \hbar^{n-1} \zeta^{(n-1)}, \end{aligned}$$

such that

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\kappa} \mathbf{y} \bmod \hbar^n, \\ \boldsymbol{\lambda} &= \mathbf{f}(\mathbf{y}) + \mathbf{K} \boldsymbol{\zeta} \bmod \hbar^n. \end{aligned}$$

– From the condition $\mathbf{f}(\mathbf{x}) = \mathbf{K}\boldsymbol{\lambda}$, we have

$$f(x^{(n)}) - Q\lambda^{(n)} = - \sum_{\ell=0}^{n-1} f^{(n-\ell)}(x^{(\ell)}) + \sum_{\ell=0}^{n-1} K^{(n-\ell)}\lambda^{(\ell)}. \quad (\text{A.53})$$

From the assumption we have, for $0 \leq \ell \leq n-1$,

$$\begin{aligned} x^{(\ell)} &= \sum_{j=1}^{\ell} \kappa^{(j)} y^{(\ell-j)}, \\ \lambda^{(\ell)} &= f(y^{(\ell)}) + \sum_{j=1}^{\ell} f^{(j)}(y^{(\ell-j)}) + Q\zeta^{(\ell)} + \sum_{j=1}^{\ell} K^{(j)}\zeta^{(\ell-j)}. \end{aligned}$$

Make the substitution as the above to (A.53), we obtain that

$$f(x^{(n)}) - Q\lambda^{(n)} = A^{(n)} + B^{(n)}, \quad (\text{A.54})$$

where

$$\begin{aligned} A^{(n)} &= - \sum_{\ell=0}^{n-1} \sum_{j=1}^{\ell} f^{(n-\ell)}(\kappa^{(j)} y^{(\ell-j)}) + \sum_{\ell=0}^{n-1} K^{(n-\ell)} f(y^{(\ell)}) + \sum_{\ell=0}^{n-1} \sum_{j=1}^{\ell} K^{(n-\ell)} f^{(j)}(y^{(\ell-j)}), \\ B^{(n)} &= + \sum_{\ell=0}^{n-1} K^{(n-\ell)} Q\zeta^{(\ell)} + \sum_{\ell=0}^{n-1} \sum_{j=1}^{\ell} K^{(n-\ell)} K^{(j)} \zeta^{(\ell-j)}. \end{aligned}$$

From $\mathbf{Kf} = \mathbf{fK}$ and $\mathbf{K}^2 = 0$, we obtain that

$$\begin{aligned} A^{(n)} &= f\left(\sum_{\ell=1}^n \kappa^{(\ell)} y^{(n-\ell)}\right) - Q \sum_{\ell=1}^n f^{(\ell)}(y^{(n-\ell)}), \\ B^{(n)} &= -Q \sum_{\ell=1}^n K^{(\ell)} \zeta^{(n-\ell)}. \end{aligned}$$

Hence (A.54) reduces to

$$f\left(x^{(n)} - \sum_{\ell=1}^n \kappa^{(\ell)} y^{(n-\ell)}\right) = Q \left(\lambda^{(n)} - \sum_{\ell=1}^n f^{(\ell)}(y^{(n-\ell)}) - \sum_{\ell=1}^n K^{(\ell)} \zeta^{(n-\ell)} \right). \quad (\text{A.55})$$

By taking the Q -cohomology class of the above relation we have

$$x^{(n)} = \sum_{\ell=1}^n \kappa^{(\ell)} y^{(n-\ell)}, \quad (\text{A.56})$$

as well as

$$Q \left(\lambda^{(n)} - \sum_{\ell=1}^n f^{(\ell)}(y^{(n-\ell)}) - \sum_{\ell=1}^n K^{(\ell)} \zeta^{(n-\ell)} \right) = 0,$$

which implies that there is a pair $\{y^{(n)}, \zeta^{(n)}\}$ such that

$$\lambda^{(n)} - \sum_{\ell=1}^n f^{(\ell)}(y^{(n-\ell)}) - \sum_{\ell=1}^n K^{(\ell)} \zeta^{(n-\ell)} = f(y^{(n)}) + Q\zeta^{(n)}.$$

Equivalently, we have

$$\lambda^{(n)} = f(y^{(n)}) + \sum_{\ell=1}^n f^{(\ell)}(y^{(n-\ell)}) + Q\zeta^{(n)} + \sum_{\ell=1}^n K^{(\ell)} \zeta^{(n-\ell)}. \quad (\text{A.57})$$

Set $\tilde{\mathbf{y}} = \mathbf{y} + \hbar^n y^{(n)}$ and $\tilde{\boldsymbol{\zeta}} = \boldsymbol{\zeta} + \hbar^n \zeta^{(n)}$. Then the relation (A.56) and (A.55) together with our assumption is equivalent to the following:

$$\begin{aligned} \tilde{\mathbf{x}} &= \boldsymbol{\kappa} \tilde{\mathbf{y}} \bmod \hbar^{n+1}, \\ \tilde{\boldsymbol{\lambda}} &= \mathbf{f}(\tilde{\mathbf{y}}) + \mathbf{K} \tilde{\boldsymbol{\zeta}} \bmod \hbar^{n+1}. \end{aligned}$$

Hence we have proved the proposition by the mathematical induction. \square

B. Appendix

Let $(\mathcal{C}, Q, (\bullet, \bullet))$ be a DG0LA over \mathbb{k} and let the cohomology H of the correspond cochain complex (\mathcal{C}, Q) is finite dimensional for each ghost number as a \mathbb{Z} -graded \mathbb{k} -vector space. Let $\{e_\alpha\}$ be a basis of H and let $t_H = \{t^\alpha\}$ be the dual basis. We denote $\{\partial_\alpha = \frac{\partial}{\partial t^\alpha}\}$ be the corresponding formal partial derivative acting on $\mathbb{k}[[t_H]]$ as derivations. Assume that we have a *fixed* versal solution $\Theta \in \mathcal{C}[[t_H]]^0$ to the Maurer-Cartan equation of the DG0LA

$$Q\Theta + \frac{1}{2}(\Theta, \Theta) = 0,$$

where

$$\Theta = t^\alpha O_\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} t^{\alpha_n} \cdots t^{\alpha_1} O_{\alpha_1 \cdots \alpha_n}$$

such that the set $\{O_\alpha\}$ from the leading term is a set of representative of the basis $\{e_\alpha\}$ of H . The Maurer-Cartan equation implies that

$$Q_\Theta := Q + (\Theta, \bullet) : \mathcal{C}[[t_H]]^i \longrightarrow \mathcal{C}[[t_H]]^{i+1}$$

satisfies $Q_\Theta^2 = 0$. By applying ∂_γ to the Maurer-Cartan equation we obtain that

$$Q\Theta_\gamma + (\Theta, \Theta_\gamma) = 0,$$

where $\Theta_\gamma := \partial_\gamma \Theta \in \mathcal{C}[[t_H]]^{|\gamma|}$.

We recall that any homogeneous element $X \in \mathcal{C}^{|X|}$ satisfying $QX = 0$ can be expressed as $X = c^\gamma O_\gamma + QY$ with unique set of constants $\{c^\gamma\}$ in \mathbb{k} and some $Y \in \mathcal{C}^{|X|-1}$ defined modulo $\text{Ker } Q$. Also for any equality in the form $QY = c^\gamma O_\gamma$ implies that $c^\gamma = 0$ for all γ , since by taking the Q -cohomology class we have $c^\gamma [O_\gamma] \equiv c^\gamma e_\gamma = 0$ and $\{e_\gamma\}$ are linearly independent, as well as that $QY = 0$. The purpose of this appendix is to confirm that the similar properties involving $Q_\Theta = Q + (\Theta, \bullet)$ and the set $\{\Theta_\gamma\}$.

Proposition B.1. *Any homogeneous element $\mathcal{X} \in \mathcal{C}[[t_H]]^{|\mathcal{X}|}$ satisfying*

$$Q_\Theta \mathcal{X} = 0$$

can be expressed as

$$\mathcal{X} = B^\gamma \Theta_\gamma + Q_\Theta \mathcal{Y},$$

where $\{B^\gamma\} \in \mathbb{k}[[t_H]]$ is defined uniquely and $\mathcal{Y} \in \mathcal{C}[[t_H]]^{|\mathcal{X}|-1}$ is defined modulo $\text{Ker } Q_\Theta$.

Proposition B.2. *Assume that we have the following equality*

$$Q_\Theta \mathcal{Y} = B^\gamma \Theta_\gamma,$$

where $\{B^\gamma\} \in \mathbb{k}[[t_H]]$. Then $B^\gamma = 0$ for all γ and $Q_\Theta \mathcal{Y} = 0$.

For our purpose we shall establish the above propositions modulo t_H^{n+1} and take the limit $n \rightarrow \infty$.

Let $\Theta \in \mathcal{C}[[t_H]]^0 \text{ mod } t_H^{n+1}$ be a versal solution of the Maurer-Cartan equation modulo t_H^{n+1} :

$$Q\Theta + \frac{1}{2}(\Theta, \Theta) = 0 \text{ mod } t_H^{n+1}.$$

Let $Q_\Theta = Q + (\Theta, \bullet) \text{ mod } t_H^{n+1}$. Then, for any $\mathcal{X} \in \mathcal{C}[[t_H]] \text{ mod } t_H^{n+1}$, we have $Q_\Theta^2 \mathcal{X} = 0 \text{ mod } t_H^{n+1}$. Let $\Theta_\gamma = \partial_\gamma \Theta \in \mathcal{C}[[t_H]]^{|\gamma|} \text{ mod } t_H^n$. Then $Q_\Theta \Theta_\gamma = 0 \text{ mod } t_H^n$. We use decompositions of $\Theta \text{ mod } t_H^{n+1}$ and $\Theta_\gamma \text{ mod } t_H^n$ in terms of the word length in t_H :

$$\begin{aligned} \Theta &= \Theta^{[1]} + \dots + \Theta^{[n]} \text{ mod } t_H^{n+1}, \\ \Theta_\gamma &= \Theta_\gamma^{[0]} + \Theta_\gamma^{[1]} + \dots + \Theta_\gamma^{[n-1]} \text{ mod } t_H^n, \end{aligned}$$

where

$$\begin{aligned} \Theta^{[k]} &= \frac{1}{k!} t^{\alpha_1} \dots t^{\alpha_k} O_{\alpha_k \dots \alpha_1} \quad \text{for } k = 1, 2, \dots, n, \\ \Theta_\gamma^{[j]} &= \frac{1}{j!} t^{\alpha_1} \dots t^{\alpha_j} O_{\alpha_j \dots \alpha_1 \gamma} \quad \text{for } j = 0, 1, \dots, n-1. \end{aligned}$$

In particular $\Theta^{[1]} = t^\alpha O_\alpha$ and $\Theta_\gamma^{[0]} = O_\gamma$, where $QO_\gamma = 0$ for all γ and the Q -cohomology classes $\{[O_\gamma]\}$ of $\{O_\gamma\}$ form a basis $\{e_\gamma\}$ of the Q -cohomology group H .

Property B.1. The condition $Q\Theta + \frac{1}{2}(\Theta, \Theta) = 0 \mod t_H^{n+1}$ is equivalent to the following sequence of equations

$$\begin{aligned} Q\Theta^{[1]} &= 0, \\ Q\Theta^{[2]} + \frac{1}{2}(\Theta^{[1]}, \Theta^{[1]}) &= 0, \\ &\vdots \\ Q\Theta^{[n]} + \frac{1}{2}\sum_{j=1}^{n-1}(\Theta^{[j]}, \Theta^{[n-j]}) &= 0. \end{aligned}$$

Property B.2. The condition that $Q\Theta_\gamma + (\Theta, \Theta_\gamma) = 0 \mod t_H^n$ is equivalent to the following sequence of equations

$$\begin{aligned} Q\Theta_\gamma^{[0]} &= 0, \\ Q\Theta_\gamma^{[1]} + (\Theta^{[1]}, \Theta_\gamma^{[0]}) &= 0, \\ &\vdots \\ Q\Theta_\gamma^{[n]} + \sum_{\ell=1}^n(\Theta^{[\ell]}, \Theta_\gamma^{[n-\ell]}) &= 0. \end{aligned}$$

Proposition B.3. Let $\Theta \in \mathcal{C}[[t_H]]^0 \mod t_H^{n+1}$ be a versal solution to the MC equation modulo t_H^{n+1} and let $\Theta_\gamma = \partial_\gamma \Theta \in \mathcal{C}[[t_H]]^{|\gamma|} \mod t_H^n$ so that $Q\Theta_\gamma + (\Theta, \Theta_\gamma) = 0 \mod t_H^n$. Then any homogeneous element $\mathcal{X} \in \mathcal{C}[[t_H]]^{|\mathcal{X}|} \mod t_H^n$ satisfying

$$Q\mathcal{X} + (\Theta, \mathcal{X}) = 0 \mod t_H^n$$

can be expressed as

$$\mathcal{X} = B^\gamma \Theta_\gamma + Q\mathcal{Y} + (\Theta, \mathcal{Y}) \mod t_H^n$$

where $B^\gamma \in \mathbb{k}[[t_H]]$ defined uniquely modulo t_H^n and $\mathcal{Y} \in \mathcal{C}[[t_H]]^{|\mathcal{X}|-1} \mod t_H^n$ defined modulo $\text{Ker } Q_\Theta$.

Proof. Decompose $\mathcal{X} \mod t_H^n$ in terms of the word length in t_H :

$$\mathcal{X} = \mathcal{X}^{[0]} + \mathcal{X}^{[1]} + \dots + \mathcal{X}^{[n-1]} \mod t_H^n.$$

Then the condition $Q\mathcal{X} + (\Theta, \mathcal{X}) = 0 \bmod t_H^n$ is equivalent to the following sequence of equations:

$$\begin{aligned} Q\mathcal{X}^{[0]} &= 0, \\ Q\mathcal{X}^{[1]} + (\Theta^{[1]}, \mathcal{X}^{[0]}) &= 0, \\ &\vdots \\ Q\mathcal{X}^{[n-1]} + \sum_{\ell=0}^{n-2} (\Theta^{[n-\ell-1]}, \mathcal{X}^{[\ell]}) &= 0. \end{aligned}$$

Then we need to find following sequence of solutions with the claimed properties:

$$\begin{aligned} \mathcal{X}^{[0]} &= B^{[0]\gamma} \Theta_\gamma^{[0]} + Q\mathcal{Y}^{[0]}, \\ \mathcal{X}^{[1]} &= B^{[0]\gamma} \Theta_\gamma^{[1]} + B^{[1]\gamma} \Theta_\gamma^{[0]} + Q\mathcal{Y}^{[1]} + (\Theta^{[0]}, \mathcal{Y}^{[1]}), \\ &\vdots \\ \mathcal{X}^{[n-1]} &= \sum_{\ell=0}^{n-1} B^{[\ell]\gamma} \Theta_\gamma^{k-\ell} + Q\mathcal{Y}^{[n-1]} + \sum_{i=1}^{n-1} (\Theta^{[i]}, \mathcal{Y}^{n-i-1}), \end{aligned}$$

for unique $B^\gamma = B^{[0]\gamma} + B^{[1]\gamma} + \dots + B^{[n-1]\gamma} \bmod t_H^n$, where $B^{[j]\gamma} = \frac{1}{j!} t^{\alpha_1} \dots t^{\alpha_j} b_{\alpha_j \dots \alpha_1}^\gamma$ and $b_{\alpha_j \dots \alpha_1}^\gamma \in \mathbb{k}$, and for some $\mathcal{Y} = \mathcal{Y}^{[0]} + \mathcal{Y}^{[1]} + \dots + \mathcal{Y}^{[n-1]} \bmod t_H^n$, where $\mathcal{Y}^{[j]} = \frac{1}{j!} t^{\alpha_1} \dots t^{\alpha_j} Y_{\alpha_j \dots \alpha_1}$ and $Y_{\alpha_j \dots \alpha_1} \in \mathcal{C}^{|\alpha_j| + \dots + |\alpha_1| - 1}$.

We are going to use mathematical induction: consider the condition $Q_\Theta \mathcal{X} = 0 \bmod t_H^1$, which is equivalent to $Q\mathcal{X}^{[0]} = 0$. Then we have

$$\mathcal{X}^{[0]} = B^{[0]\gamma} \Theta_\gamma^{[0]} + Q\mathcal{Y}^{[0]}$$

for unique $B^{[0]\gamma} \in \mathbb{k}$ and for some $\mathcal{Y}^{[0]} \in \mathcal{C}^{|\mathcal{X}|}$ defined modulo $\text{Ker } Q$, since $\mathcal{X}^{[0]} \in \mathcal{C}$ and $\Theta_\gamma^{[0]} = O_\gamma$. Hence our claim is true modulo t_H^1 .

Fix k such that $1 < k < n - 1$ and assume that our claim is true for modulo t_H^{k+1} :

$$\mathcal{X} = B^\gamma \Theta_\gamma + Q\mathcal{Y} + (\Theta, \mathcal{Y}) \bmod t_H^{k+1} \quad (6.1)$$

where $B^\gamma = B^{[0]\gamma} + B^{[1]\gamma} \dots + B^{[k]\gamma} \bmod t_H^{k+1}$ defined uniquely and $\mathcal{Y} = \mathcal{Y}^{[0]} + \mathcal{Y}^{[1]} + \dots + \mathcal{Y}^{[k]} \bmod t_H^{k+1}$ defined modulo $\text{Ker } Q_\Theta$. We have, in components, for all $j = 0, 1, \dots, k$,

$$\mathcal{X}^{[j]} = \sum_{\ell=0}^j B^{[\ell]\gamma} \Theta_\gamma^{j-\ell} + Q\mathcal{Y}^{[j]} + \sum_{i=1}^j (\Theta^{[i]}, \mathcal{Y}^{[j-i]}).$$

Then, we need to show that there exist unique $B^{[k+1]\gamma}$ and $\mathcal{Y}^{[k+1]}$ satisfying

$$\mathcal{X}^{[k+1]} = \sum_{\ell=0}^{k+1} B^{[\ell]\gamma} \Theta_{\gamma}^{k+1-\ell} + Q\mathcal{Y}^{[k+1]} + \sum_{i=1}^{k+1} (\Theta^{[i]}, \mathcal{Y}^{[k+1-i]}),$$

such that

$$\mathcal{X} = B^{\gamma} \Theta_{\gamma} + Q\mathcal{Y} + (\Theta, \mathcal{Y}) \bmod t_H^{k+2}$$

for unique $B^{\gamma} = B^{[0]\gamma} + B^{[1]\gamma} + \dots + B^{[k]\gamma} + B^{[k+1]\gamma} \bmod t_H^{k+2}$ and for some $\mathcal{Y} = \mathcal{Y}^{[0]} + \mathcal{Y}^{[1]} + \dots + \mathcal{Y}^{[k]} + \mathcal{Y}^{[k+1]} \bmod t_H^{k+2}$ is defined modulo $\text{Ker } Q_{\Theta}$.

We shall proceed as follows:

- (Claim 1): we claim that the following expression:

$$\mathcal{J}^{[k+1]} := \mathcal{X}^{[k+1]} - \sum_{\ell=1}^{k+1} B^{[k+1-\ell]\gamma} \Theta_{\gamma}^{[\ell]} - \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{Y}^{[k+1-\ell]})$$

satisfies $Q\mathcal{J}^{[k+1]} = 0$.

- (Claim 2): Note that the last term of the expression $\mathcal{J}^{[k+1]}$ is ambiguous due to the ambiguities of $\mathcal{Y} = \mathcal{Y}^{[0]} + \dots + \mathcal{Y}^{[k]} \bmod t_H^{k+1}$, which is assumed to be defined modulo $\text{Ker } Q + (\Theta, \cdot)$. We claim that the resulting ambiguity of $\mathcal{J}^{[k+1]}$ is Q -exact. That is, if $\mathcal{Y}' = \mathcal{Y}'^{[0]} + \dots + \mathcal{Y}'^{[k]} \bmod t_H^{k+1}$ denote any other possible choice for (6.1), and let

$$\mathcal{J}'^{[k+1]} := \mathcal{X}^{[k+1]} - \sum_{\ell=1}^{k+1} B^{[k+1-\ell]\gamma} \Theta_{\gamma}^{[\ell]} - \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{Y}'^{[k+1-\ell]}),$$

then

$$\mathcal{J}^{[k+1]} - \mathcal{J}'^{[k+1]} = \sum_{\ell=0}^k (\Theta^{[k+1-\ell]}, \mathcal{Y}'^{[\ell]} - \mathcal{Y}^{[\ell]}) \in \text{Im } Q.$$

Hence both $\mathcal{J}^{[k+1]}$ and $\mathcal{J}'^{[k+1]}$ belong to the same Q -cohomology class.

From the above claims, it follows that there is unique set $\{B^{[k+1]\gamma}\}$, where $B^{[k+1]\gamma} = \frac{1}{(k+1)!} t^{\rho_1} \dots t^{\rho_{k+1}} b_{\rho_{k+1} \dots \rho_1}^{\gamma}$, such that

$$\begin{aligned} \mathcal{J}^{[k+1]} &= B^{[k+1]\gamma} \Theta_{\gamma}^{[0]} + Q\mathcal{Y}^{[k+1]}, \\ \mathcal{J}'^{[k+1]} &= B^{[k+1]\gamma} \Theta_{\gamma}^{[0]} + Q\mathcal{Y}'^{[k+1]}, \end{aligned} \tag{6.2}$$

for some $\mathcal{Y}^{[k+1]}, \mathcal{Y}'^{[k+1]}$ defined modulo $\text{Ker } Q$. From the above and the definitions of $\mathcal{J}^{[k+1]}$ and $\mathcal{J}'^{[k+1]}$ we obtain that

$$\begin{aligned}\mathcal{X}^{[k+1]} &= \sum_{\ell=0}^{k+1} B^{[k+1-\ell]\gamma} \Theta_{\gamma}^{[\ell]} + Q\mathcal{Y}^{[k+1]} + \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{Y}^{[k+1-\ell]}) \\ &= \sum_{\ell=0}^{k+1} B^{[k+1-\ell]\gamma} \Theta_{\gamma}^{[\ell]} + Q\mathcal{Y}'^{[k+1]} + \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{Y}'^{[k+1-\ell]}).\end{aligned}\quad (6.3)$$

From eq. (6.2) we also obtain that $\mathcal{J}^{[k+1]} - Q\mathcal{Y}^{[k+1]} = \mathcal{J}'^{[k+1]} - Q\mathcal{Y}'^{[k+1]}$, which implies that

$$Q(\mathcal{Y}'^{[k+1]} - \mathcal{Y}^{[k+1]}) + \sum_{\ell=0}^k (\Theta^{[k+1-\ell]}, \mathcal{Y}'^{[\ell]} - \mathcal{Y}^{[\ell]}) = 0. \quad (6.4)$$

Let $\mathcal{Y} := \mathcal{Y}^{[0]} + \dots + \mathcal{Y}^{[k]} + \mathcal{Y}^{[k+1]} \bmod t_H^{[k+2]}$ and $\mathcal{Y}' := \mathcal{Y}'^{[0]} + \dots + \mathcal{Y}'^{[k]} + \mathcal{Y}'^{[k+1]} \bmod t_H^{[k+2]}$. Combined with our assumption, the relations in eq. (6.3) imply that

$$\begin{aligned}\mathcal{X} &= B^{\gamma} \Theta_{\gamma} + Q\mathcal{Y} + (\Theta, \mathcal{Y}) \bmod t_H^{k+2} \\ &= B^{\gamma} \Theta_{\gamma} + Q\mathcal{Y}' + (\Theta, \mathcal{Y}') \bmod t_H^{k+2}.\end{aligned}$$

Combined with our assumption, the relation in eq. (6.4) implies that

$$Q(\mathcal{Y}' - \mathcal{Y}) + (\Theta, \mathcal{Y}' - \mathcal{Y}) = 0 \bmod t_H^{k+2}.$$

Thus, we shall have a proof of our proposition by mathematical induction the two claims above are checked.

Proof of Claim 1. We shall show that

$$\sum_{j=0}^k (\Theta^{[k+1-j]}, \mathcal{X}^{[j]}) = -Q \left(\sum_{\ell=1}^{k+1} B^{[k+1-\ell]\gamma} \Theta_{\gamma}^{[\ell]} + \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{Y}^{[k+1-\ell]}) \right) .. \quad (6.5)$$

From the condition $Q\mathcal{X} + (\Theta, \mathcal{X}) = 0 \bmod t_H^{n+1}$, which implies that

$$Q\mathcal{X}^{[k+1]} + \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{X}^{[k+1-\ell]}) = 0,$$

we see that (6.5) is equivalent to the claim that $Q\mathcal{J}^{[k+1]} = 0$.

Consider the left hand side of eq. (6.5):

$$\sum_{j=0}^k (\Theta^{[k+1-j]}, \mathcal{X}^{[j]}).$$

From the assumption (6.1) we have

$$\begin{aligned} \sum_{j=0}^k (\Theta^{[k+1-j]}, \mathcal{X}^{[j]}) &= \sum_{j=0}^k \sum_{\ell=0}^j B^{[\ell]\gamma} (\Theta^{[k+1-j]}, \Theta_\gamma^{[j-\ell]}) \\ &\quad + \sum_{j=0}^k (\Theta^{[k+1-j]}, Q\mathcal{Y}^{[j]}) + \sum_{j=0}^k \sum_{i=1}^j (\Theta^{[k+1-j]}, (\Theta^{[i]}, \mathcal{Y}^{[j-i]})). \end{aligned}$$

We consider the first and second lines in the right hand side of the above separately:

- For the first line we have

$$\begin{aligned} \sum_{j=0}^k \sum_{\ell=0}^j B^{[\ell]\gamma} (\Theta^{[k+1-j]}, \Theta_\gamma^{[j-\ell]}) &= \sum_{\ell=1}^{k+1} B^{[k+1-\ell]\gamma} \left(\sum_{i=1}^{\ell} (\Theta^{[i]}, \Theta_\gamma^{[\ell-i]}) \right) \\ &= -Q \left(\sum_{\ell=1}^{k+1} B^{[k+1-\ell]\gamma} \Theta_\gamma^{[\ell]} \right) \end{aligned}$$

where we did re-summations for the first equality and used Property B.2 for the second equality.

- For the second line we have

$$\begin{aligned} \sum_{j=0}^k (\Theta^{[k+1-j]}, Q\mathcal{Y}^{[j]}) &+ \sum_{j=0}^k \sum_{i=1}^j (\Theta^{[k+1-j]}, (\Theta^{[i]}, \mathcal{Y}^{[j-i]})) \\ &= -Q \sum_{j=0}^k (\Theta^{[k+1-j]}, \mathcal{Y}^{[j]}) \\ &\quad + \sum_{j=0}^k (Q\Theta^{[k+1-j]}, \mathcal{Y}^{[j]}) + \sum_{j=0}^k \sum_{i=1}^j (\Theta^{[k+1-j]}, (\Theta^{[i]}, \mathcal{Y}^{[j-i]})) \\ &= -Q \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{Y}^{[k+1-\ell]}) \\ &\quad + \sum_{j=1}^{k+1} \left(Q\Theta^{[j]} + \sum_{i=1}^{j-1} (\Theta^{[i]}, \Theta^{[j-i]}) \right), \mathcal{Y}^{[k+1-j]} \\ &= -Q \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{Y}^{[k+1-\ell]}), \end{aligned}$$

where we have used the property that Q is a derivation of the BV bracket in the first equality, re-summations in the second equality and the Jacobi identity of the BV bracket in the third equality, and the MC equation modulo t_H^{n+1} (Property B.1) for the last equality. Thus we have established the identity in eq. (6.5).

Proof of Claim 2. Consider the last term of $\mathcal{J}^{[k+1]}$ defined in claim 1, which can be re-summed as

$$-\sum_{\ell=0}^k (\Theta^{[k+1-\ell]}, \mathcal{Y}^{[\ell]})$$

Let $\mathcal{Y} = \mathcal{Y}^{[0]} + \dots + \mathcal{Y}^{[k]} \bmod t_H^{k+1}$. By the assumption that $\mathcal{Y} \bmod t_H^{k+1}$ is defined modulo $\text{Ker } Q + (\Theta, \cdot)$. Let $\mathcal{Y}' = \mathcal{Y}'^{[0]} + \dots + \mathcal{Y}'^{[k]} \bmod t_H^{k+1}$ denote any other possible choice such that $(\mathcal{Y}' - \mathcal{Y}) \bmod t_H^{k+1}$ belongs to $\text{Ker } Q + (\Theta, \cdot)$. Then, by the assumption, we have

$$\mathcal{Y}' - \mathcal{Y} = C^\gamma \Theta_\gamma + Q\mathcal{W} + (\Theta, \mathcal{W}) \bmod t_H^{k+1}$$

for some $C^\gamma \in \mathbb{K}[[t_H]]/t_H^{k+1}$ and some $\mathcal{W} \in (\mathcal{C} \otimes \mathbb{K}[[t_H]]/t_H^{k+1})^{|\mathcal{Y}|-1}$. Thus for each $0 \leq \ell \leq k$, we have

$$\mathcal{Y}^{[\ell]'} - \mathcal{Y}^{[\ell]} = (Y^\gamma \Theta_\gamma)^{[\ell]} + Q\mathcal{W}^{[\ell]} + (\Theta, \mathcal{W})^{[\ell]}.$$

Let

$$\mathcal{J}'^{[k+1]} := \mathcal{X}^{[k+1]} - \sum_{\ell=1}^{k+1} B^{[k+1-\ell]\gamma} \Theta_\gamma^{[\ell]} - \sum_{\ell=1}^{k+1} (\Theta^{[\ell]}, \mathcal{Y}'^{[k+1-\ell]}),$$

then

$$\begin{aligned} \mathcal{J}^{[k+1]} - \mathcal{J}'^{[k+1]} &= \sum_{\ell=0}^k (\Theta^{[k+1-\ell]}, \mathcal{Y}'^{[\ell]} - \mathcal{Y}^{[\ell]}) \\ &= (\Theta, Y^\gamma \Theta_\gamma)^{[k+1]} + (\Theta, Q\mathcal{W} + (\Theta, \mathcal{W}))^{[k+1]}. \end{aligned}$$

It remains to show that the right hand side of the above is Q -exact. The first term can be rearranged as follows

$$(\Theta, Y^\gamma \Theta_\gamma)^{[k+1]} = \sum_{j=0}^k Y^{[j]\gamma} (\Theta, \Theta_\gamma)^{[k+1-j]}.$$

We then use the identity $Q\Theta + (\Theta, \Theta_\gamma) = 0 \bmod t_H^n$ (Property B.2) to obtain that

$$(\Theta, Y^\gamma \Theta_\gamma)^{[k+1]} = -Q \sum_{j=0}^k Y^{[j]\gamma} \Theta_\gamma^{[k+1-j]}.$$

For the second term, we have

$$\begin{aligned} (\Theta, Q\mathcal{W} + (\Theta, \mathcal{W}))^{[k+1]} &= -Q((\Theta, \mathcal{W})^{[k+1]} + \left(Q\Theta + \frac{1}{2}(\Theta, \Theta), \mathcal{W}\right)^{[k+1]}) \\ &= -Q(\Theta, \mathcal{W})^{[k+1]} \end{aligned}$$

where we used the property that Q is a derivation of the BV bracket and the Jacobi identity of the BV bracket for the first equality and the MC equation $Q\Theta + \frac{1}{2}(\Theta, \Theta) = 0 \bmod t_H^{n+1}$ for the second equality. Combined together we have

$$\begin{aligned} \mathcal{J}^{[k+1]} - \mathcal{J}'^{[k+1]} &= \sum_{\ell=0}^k (\Theta^{[k+1-\ell]}, \mathcal{Y}'^{[\ell]} - \mathcal{Y}^{[\ell]}) \\ &= -Q \left(\sum_{j=0}^k Y^{[j]\gamma} \Theta_\gamma^{[k+1-j]} + (\Theta, \mathcal{W})^{[k+1]} \right), \end{aligned}$$

to establish the claim. \square

Proposition B.4. *Let $\Theta \in \mathcal{C}[[t_H]]^0 \bmod t_H^{n+1}$ be a versal solution to the MC equation modulo t_H^{n+1} and let $\Theta_\gamma = \partial_\gamma \Theta \in \mathcal{C}[[t_H]]^{|\gamma|} \bmod t_H^n$. Let $m \leq n$. Assume that we have the following equality*

$$Q\mathcal{X} + (\Theta, \mathcal{X}) = C^\gamma \Theta_\gamma \bmod t_H^m$$

where $\mathcal{X} \in \mathcal{C}[[t_H]]^{|\mathcal{X}|} \bmod t_H^m$ and $C^\gamma \in \mathbb{K}[[t_H]] \bmod t_H^m$. Then

$$\begin{aligned} C^\gamma &= 0 \bmod t_H^m \quad \forall \gamma, \\ Q\mathcal{X} + (\Theta, \mathcal{X}) &= 0 \bmod t_H^m. \end{aligned}$$

and

Proof. It is obvious, for the case $m = 1$, since the condition $Q\mathcal{X} + (\Theta, \mathcal{X}) = C^\gamma \Theta_\gamma \bmod t_H^1$ is equivalent to

$$Q\mathcal{X}^{[0]} = C^{[0]\gamma} \Theta_\gamma^{[0]} = C^{[0]\gamma} \Theta_\gamma$$

so that $C^{[0]\gamma} = 0$ for all γ and $Q\mathcal{X}^{[0]} = 0$. It follows that

$$\mathcal{X}^{[0]} = B^{[0]\gamma} \Theta_\gamma^{[0]} + Q\mathcal{Y}^{[0]}.$$

Fix j such that $2 \leq j \leq m - 1$ and assume that

$$\begin{aligned} Q\mathcal{X} + (\Theta, \mathcal{X}) &= 0 \bmod t_H^j, \\ C^\gamma &= 0 \bmod t_H^j \quad \forall \gamma. \end{aligned} \tag{6.6}$$

Then the condition

$$Q\mathcal{X} + (\Theta, \mathcal{X}) = C^\gamma \Theta_\gamma \bmod t_H^{j+1}$$

is equivalent to

$$Q\mathcal{X}^{[j+1]} + (\Theta, \mathcal{X})^{[j+1]} = C^{[j+1]\gamma} \Theta_\gamma^{[0]}, \tag{6.7}$$

in addition to (6.6), and we need to show that $C^{[j+1]\gamma} = 0$ for all γ .

From proposition B.3, the assumption that $Q\mathcal{X} + (\Theta, \mathcal{X}) = 0 \bmod t_H^j$ implies that

$$\mathcal{X} = B^\gamma \Theta_\gamma + Q\mathcal{Y} + (\Theta, \mathcal{Y}) \bmod t_H^j$$

with some $B^\gamma \in \mathbb{k}[[t_H]] \bmod t_H^j$ and some $\mathcal{Y} \in \mathcal{C}[[t_H]]^{|\mathcal{X}|-1} \bmod t_H^j$. Then the term $(\Theta, \mathcal{X})^{[j+1]}$ in (6.7) becomes

$$\begin{aligned} (\Theta, \mathcal{X})^{[j+1]} &= (\Theta, B^\gamma \Theta_\gamma + Q\mathcal{Y} + (\Theta, \mathcal{Y}))^{[j+1]} \\ &= (\Theta, B^\gamma \Theta_\gamma)^{[j+1]} - Q(\Theta, \mathcal{Y})^{[j+1]} + (Q\Theta, \mathcal{Y})^{[j+1]} + (\Theta, (\Theta, \mathcal{Y}))^{[j+1]} \\ &= -Q(B^\gamma \Theta_\gamma + (\Theta, \mathcal{Y}))^{[j+1]}, \end{aligned}$$

where we have used the property that Q is a derivation of the bracket, i.e.,

$$Q(\Theta, \mathcal{Y})^{[j+1]} = (Q\Theta, \mathcal{Y})^{[j+1]} - (\Theta, Q\mathcal{Y})^{[j+1]},$$

for the 2nd equality and the assumptions that $Q\Theta_\gamma + (\Theta, \Theta_\gamma) = 0 \bmod t_H^n$ and $Q\Theta + \frac{1}{2}(\Theta, \Theta) = 0 \bmod t_H^{n+1}$ after the Jacobi law of the bracket for the last equality. Hence (6.7) implies that

$$Q(\mathcal{X} - B^\gamma \Theta_\gamma - (\Theta, \mathcal{Y}))^{[j+1]} = C^{[j+1]\gamma} \Theta_\gamma^{[0]}.$$

It follows that $C^{[j+1]\gamma} = 0$ for $\forall \gamma$ and $Q\mathcal{X}^{[j+1]} + (\Theta, \mathcal{X})^{[j+1]} = 0$. Thus we have established that $C^\gamma = 0 \bmod t_H^{j+1}$ for all γ and $Q\mathcal{X} + (\Theta, \mathcal{X}) = 0 \bmod t_H^{j+1}$. Our proposition follows by mathematical induction. \square

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